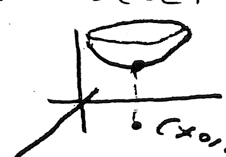
## 15.1 Local Extrema of a Function $z = f(x,y)$

<u>Def</u>: (a) $f(x,y)$ has a <u>local minimum</u> at $(x_0, y_0)$ if for all $(x,y)$ in some open disk around $(x_0,y_0)$ we have:

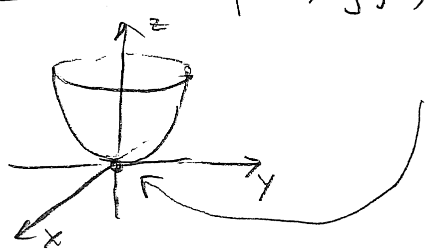
$$f(x,y) \geq f(x_0,y_0).$$




(b) $f(x,y)$ has a <u>local maximum</u> at $(x_0,y_0)$ if for all $(x,y)$ in some open disk around $(x_0, y_0)$ we have

$$f(x,y) \leq f(x_0,y_0).$$


<u>Ex</u>: $z = f(x,y)$, $f(x,y) = x^2 + y^2$.




$(0,0)$ a local minimum (and global minimum).

If $(x_0, y_0)$ is a local extremum of $f(x,y)$, then both cross-sections $f(x, y_0)$ and $f(x_0, y)$ have a local extremum at $x_0$ and $y_0$, respectively. Thus, their derivatives are 0 at at $x_0$ and $y_0$, respectively or the derivatives do not exist. Thus:

<u>Th</u>: If $f(x,y)$ has a local extremum at $(x_0, y_0)$, then

$f_x(x_0, y_0) = 0$ or $f_x(x_0,y_0)$ does not exist and

$f_y(x_0,y_0) = 0$ or $f_y(x_0,y_0)$ does not exist.

Note if the tangent plane at $(x_0, y_0, f(x_0, y_0))$ exists and $(x_0, y_0)$ is a local extremum, the tangent plane must be horizontal $f_x(x_0, y_0) = f_y(x_0, y_0) = 0$.

Def: $(x_0, y_0)$ is a critical point of $f(x,y)$ if

$f_x(x_0, y_0) = f_y(x_0, y_0) = 0$ or at least one of $f_x(x_0, y_0)$, $f_y(x_0, y_0)$ does not exist.

Th: If $f(x,y)$ has a local extremum at $(x_0, y_0)$, then $(x_0, y_0)$ is a critical point.

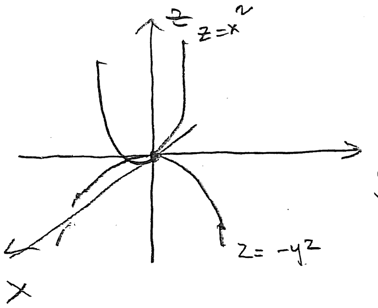
Note if $(x_0, y_0)$ is a critical point and both partials exist, then they are both 0. Thus $\nabla f(x_0, y_0) = \vec{0}$. The gradient is the zero vector at a point of local extremum.

Question: Let $z = f(x,y)$, $(x_0, y_0)$ be a critical point of $f(x,y)$. Does $(x_0, y_0)$ have to be a local extremum?

No: Consider

Ex: $f(x,y) = x^2 - y^2$. $f_x(x,y) = 2x$, $f_y(x,y) = 2y$

At $(x_0, y_0) = (0, 0)$ (and only there) both partials are 0, so $(0, 0)$ is a critical point. Do we have a local minimum or maximum at $(0,0)$?




$f(x,y) = x^2 - y^2$

Cross-section with $x=0$:

$f(0,y) = -y^2$

Cross-section with $y=0$:

$f(x,y) = x^2$

So $f(0,0)=0$ is neither the smallest not the largest value no matter how small disk about $(0,0)$ we take.

The graph is saddle-shaped, the contour diagram shows the characteristic saddle behavior:

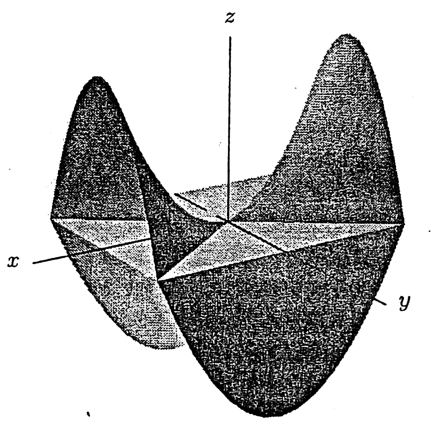



*Figure 8.55*: Graph of $f(x,y) = x^2 - y^2$ showing plane $z = 0$

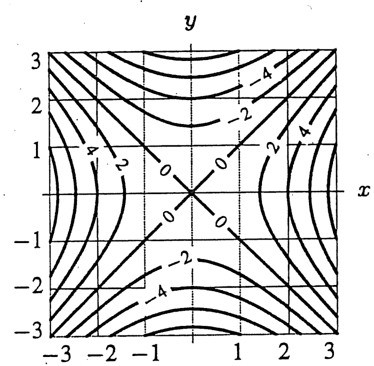


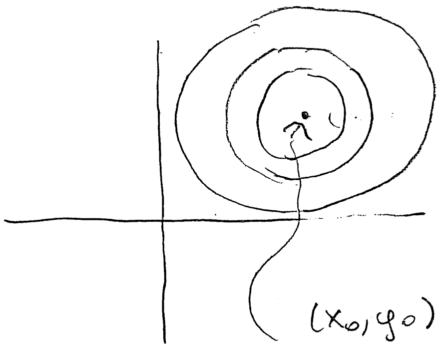

*Figure 8.54*: Contour map of $f(x,y) = x^2 - y^2$

<u>Def</u>: $(x_0, y_0)$ is a <u>saddle point</u> of $f(x,y)$ if $(x_0, y_0)$ is a critical point of $f(x,y)$ and in every open disk around $(x_0, y_0)$ there exist points $(x_1, y_1)$, $(x_2, y_2)$ such that:

$$f(x_1, y_1) < f(x_0, y_0), \quad f(x_2, y_2) > f(x_0, y_0).$$

<u>Remark</u>: If $f(x,y)$ has a critical point at $(x_0, y_0)$, then $(x_0, y_0)$ is a local minimum, a local maximum or a saddle point.

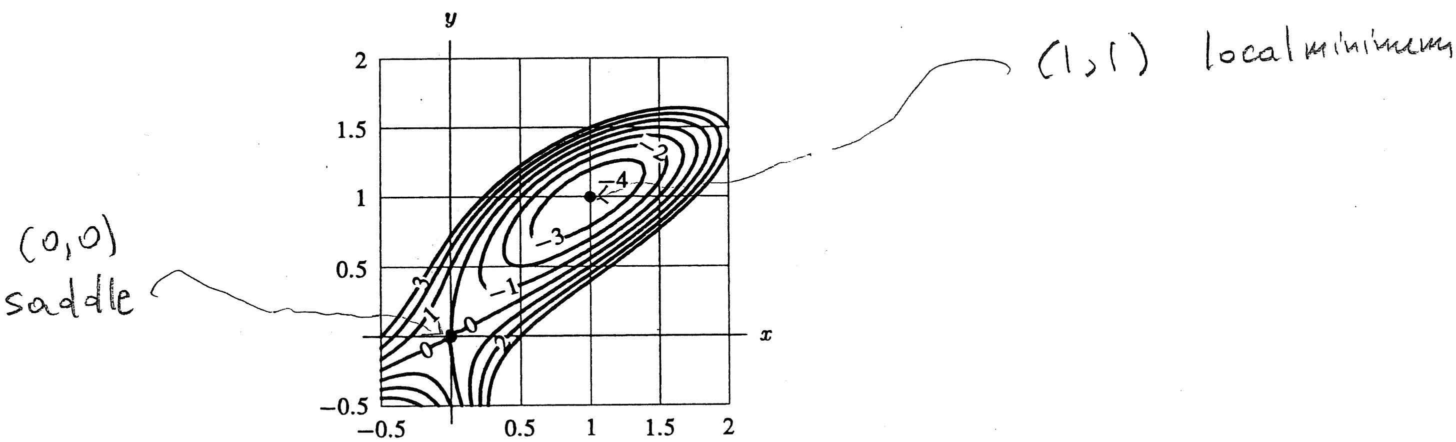
We see above a typical appearance of a contour diagram around a saddle point. If $(x_0, y_0)$ is a local minimum or maximum, the contour diagram looks something like:




Values of contours <u>increase</u> when you move away from $(x_0, y_0)$ if $(x_0, y_0)$ is a <u>minimum</u>.

Values of contours <u>decrease</u> when we move away from $(x_0, y_0)$ if $(x_0, y_0)$ is a maximum.

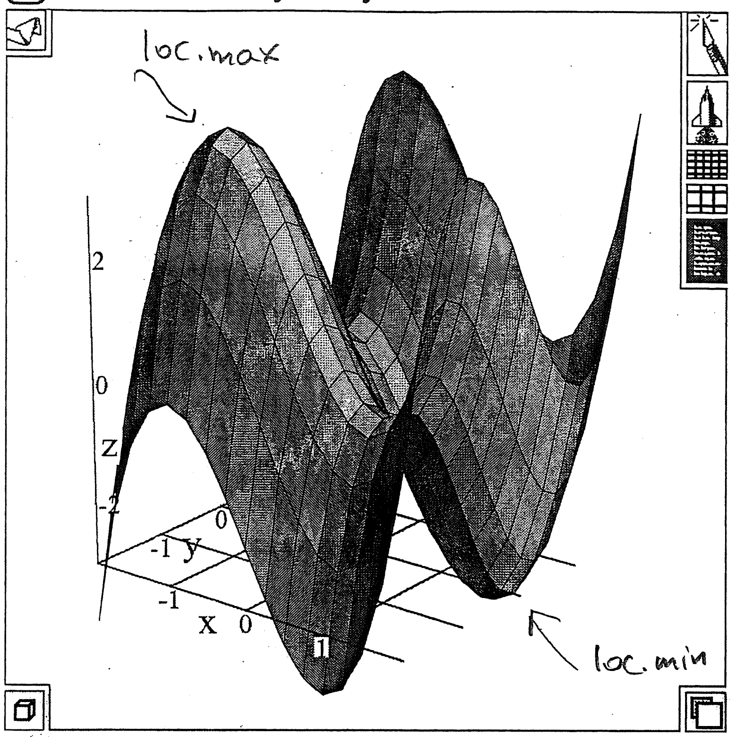
<u>Ex</u>: If the contour map of $f(x, y)$ is:




Where do you see likely critical points? Are they minima, maxima or saddle points?

Ex:

$z = x^3 - 3x + y^3 - 3y$

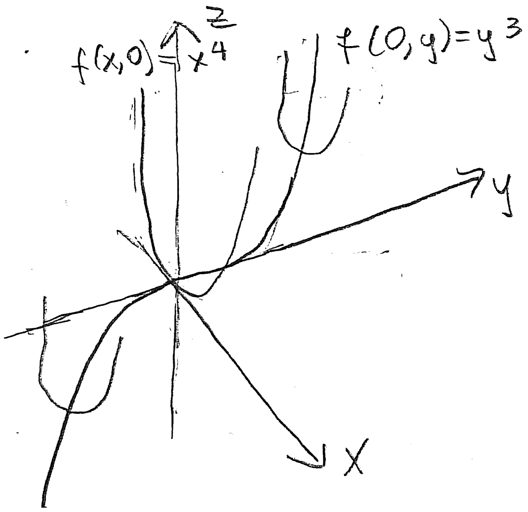



Two saddle points that we can see.

Note: The graph around a saddle point doesn't have to look like a "riding saddle".

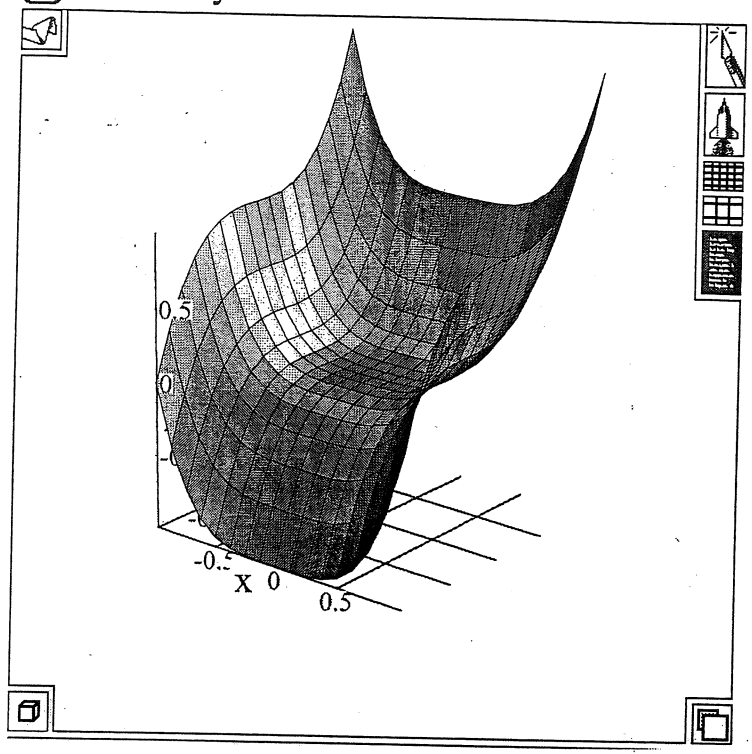
Ex: $f(x,y) = x^4 + y^3$

$f_x = 4x^3$, $f_y = 3y^2$, $(0,0)$ the only critical point.




A saddle point as $f(0,y) = y^3$ takes positive and negative values in any disk about $(0,0)$.

$z = x^4 + y^3$




A "chair".

Is a given critical point a minimum a maximum or a saddle point? The key is the discriminant and the Second Derivative Test.

Let $f(x,y)$ be given and have continuous second partials. We define the discriminant of $f(x,y)$ to be:

$$D(x,y) = f_{xx}(x,y) \cdot f_{yy}(x,y) - f_{xy}(x,y)^2$$

**Second Derivative Test for Functions of Two Variables**

Suppose $(x_0, y_0)$ is a point where $f_x(x_0, y_0) = f_y(x_0, y_0) = 0$. Let

$$D = f_{xx}(x_0, y_0) f_{yy}(x_0, y_0) - f_{xy}(x_0, y_0)^2.$$

- If $D > 0$ and $f_{xx}(x_0, y_0) > 0$, then $f$ has a local minimum at $(x_0, y_0)$.
- If $D > 0$ and $f_{xx}(x_0, y_0) < 0$, then $f$ has a local maximum at $(x_0, y_0)$.
- If $D < 0$, then $f$ has a saddle point at $(x_0, y_0)$.
- If $D = 0$, anything can happen at $(x_0, y_0)$.

Ex: $f(x,y) = x^3 - 3x + y^3 - 3y$

$f_x = 3x^2 - 3$ , $f_y = 3y^2 - 3$

Crt. pts:

$\begin{cases} 3x^2 - 3 = 0 \\ 3y^2 - 3 = 0 \end{cases}$ $x^2 = 1$ and $y^2 = 1$

Crt. pts: $(-1,-1), (-1,1), (1,-1), (1,1)$.

$f_{xx} = 6x$ , $f_{yy} = 6y$ , $f_{xy} = 0$

$D(x,y) = 36xy$

$D(1,1) > 0$ , $f_{xx}(1,1) > 0$ – loc. min

$D(1,-1) < 0$ , $D(-1,1) < 0$ – saddle points

$D(-1,-1) > 0$ , $f_{xx}(-1,-1) < 0$ – loc. max.

Look at the graph on page 13.