14.7 Second Partials

Given $z = f(x,y)$ each partial derivative

$$\frac{\partial z}{\partial x} = f_x(x,y) \quad , \quad \frac{\partial z}{\partial y} = f_y(x,y)$$

is again a function of (x,y). So we can take their partial derivatives. Partial derivatives of partial derivatives are called second partial derivatives. We denote them:

$$(f_x)_x = f_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}[f_x(x,y)]$$

$$(f_x)_y = f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}[f_x(x,y)]$$

first second   second first

$$(f_y)_x = f_{yx} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x}[f_y(x,y)]$$

$$(f_y)_y = f_{yy} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}[f_y(x,y)].$$

Four partials

$$f_{xy} \ , \ f_{yx}$$

are called mixed partials.

Ex: $z = f(x,y)$, $f(x,y) = x^3 + x^2y^2 - y^4$.

Find second partials.

$f_x = 3x^2 + 2xy^2$, $f_y = 2x^2y - 4y^3$

$f_{xx} = 6x + 2y^2$, $f_{xy} = 4xy$

$f_{yy} = 2x^2 - 12y^2$, $f_{yx} = 4xy$

$f_{xy} = f_{yx}$

Is it always so?

Th: If $f_{xy}(x,y)$ and $f_{yx}(x,y)$ are continuous, then $f_{xy}(x,y) = f_{yx}(x,y)$.

Second partials important, among others, in the context of

- Taylor approximations of order 2
- Second Derivative Test for Local Extrema

Remark: We are not going to talk much about conditions for differentiability of a function $z = f(x,y)$ but let us state at least the following:

Th: If $f_x(x,y)$, $f_y(x,y)$ exist and are continuous in a disk centered at (a,b), then $f(x,y)$ is differentiable at (a,b).