March 2, 4

14.6 The Chain Rule

More precisely: the chain rules. For functions of several variables there are many versions of the chain rule depending on what is a function of what. To get the idea, let's have a few special cases.

Case 1: Let $z = f(x,y)$, $x = g(t)$, $y = h(t)$.

So z is a function of x, y, and each x and y are functions of some variable t. Then z is a function of t:

$$z = f(g(t), h(t)),$$

What is $\frac{dz}{dt}$? If f, g, and h are all differentiable we have:

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}.$$

If we want $\left.\frac{dz}{dt}\right|_{t=p}$, denote $(a,b) = (x(p), y(p))$:

$$\left.\frac{dz}{dt}\right|_{t=p} = \left.\frac{\partial z}{\partial x}\right|_{(a,b)} \cdot \left.\frac{dx}{dt}\right|_{p} + \left.\frac{\partial z}{\partial y}\right|_{(a,b)} \cdot \left.\frac{dy}{dt}\right|_{p}$$

Why? Here is some justification. We want to convince ourselves that

$$\left.\frac{dz}{dt}\right|_{t=p} = f_x(a,b)\cdot g'(p) + f_y(a,b)\cdot h'(p) \qquad (*)$$

We have

$$\left.\frac{dz}{dt}\right|_{t=p} = \lim_{\Delta t\to 0}\frac{\Delta z}{\Delta t} \qquad \text{so} \qquad \left.\frac{dz}{dt}\right|_{t=p} \cong \frac{\Delta z}{\Delta t}$$

How can we estimate:

$$\frac{\Delta z}{\Delta t} = \frac{\Delta f}{\Delta t} \ ?$$

A change Δt produces changes Δx and Δy in x and y where:

$$\Delta x = g(p+\Delta t) - g(p) \cong g'(p)\cdot\Delta t, \quad \Delta y = h(p+\Delta t) - h(p) \cong h'(p)\cdot\Delta t$$

Changes Δx and Δy produce a change in f approximately equal:

$$\Delta f = f(a+\Delta x, b+\Delta y) - f(a,b) \cong f_x(a,b)\cdot\Delta x + f_y(a,b)\cdot\Delta y =$$

$$\cong f_x(a,b)\cdot g'(p)\cdot\Delta t + f_y(a,b)\, h'(p)\cdot\Delta t$$

So

$$\frac{\Delta f}{\Delta t} \cong f_x(a,b)\cdot g'(p) + f_y(a,b)\cdot h'(p)$$

This gives us some hint why $(*)$ is true.

Ex: Let $z = f(x,y) = x\sin(y)$, $x = t^2$, $y = 2t+1$.

Find

(a) $\frac{dz}{dt}$ (b) $\left.\frac{dz}{dt}\right|_{t=0}$.

We use the chain rule:

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dt} = \sin(y)\cdot 2t + x\cos(y)\cdot 2 =$$

$$= \sin(2t+1)\cdot 2t + t^2\cos(2t+1)\cdot 2$$

$$\left.\frac{dz}{dt}\right|_{t=0} = 0.$$

Or we can leave:

$$\frac{dz}{dt} = \sin(y)\cdot 2t + x\cos(y)\cdot 2$$

and reason as follows. When $t=0$, $x=0$, $y=1$. So

$$\left.\frac{dz}{dt}\right|_{t=0} = \sin(1)\cdot 0 + 0\cdot\cos(1)\cdot 2 = 0.$$

A more interesting problem:

Ex: Let $z = (x+y)e^y$. Suppose $x = g(t)$, $y = h(t)$. Given that $g(0) = 2$, $h(0) = 0$, $g'(0) = -0.3$, $h'(0) = 2.5$, find $\frac{dz}{dt}\Big|_{t=0}$

When $t = 0$, $x = 2$ and $y = 0$; So

$$\frac{dz}{dt}\Big|_{t=0} = \frac{\partial z}{\partial x}\Big|_{(2,0)} \cdot \frac{dx}{dt}\Big|_{t=0} + \frac{\partial z}{\partial y}\Big|_{(2,0)} \cdot \frac{dy}{dt}\Big|_{t=0} =$$

$$= \frac{\partial z}{\partial x}\Big|_{(2,0)} \cdot g'(0) + \frac{\partial z}{\partial y}\Big|_{(2,0)} \cdot h'(0)$$

$$\frac{\partial z}{\partial x}\Big|_{(2,0)} = e^y\Big|_{(2,0)} = 1, \quad \frac{\partial z}{\partial y}\Big|_{(2,0)} = e^y + (x+y)e^y\Big|_{(2,0)} = 3$$

Thus:

$$\frac{dz}{dt}\Big|_{t=0} = 1 \cdot (-0.3) + 3 \cdot 2.5 = 7.2$$

<u>Case 1a</u>: $w = f(x, y, z)$, $x = x(t)$, $y = y(t)$, $z = z(t)$.

Then:

$$\frac{dw}{dt} = \frac{\partial w}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial w}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial w}{\partial z} \cdot \frac{dz}{dt}.$$

The book gives a mnemotechnical way of obtaining versions of the chain rule in all possible cases. Draw a diagram containing all the variables and edges which represent which variable depends on which. For example:

$$\begin{tikzcd}
 & w \arrow[dl, dash, "\frac{\partial w}{\partial x}"'] \arrow[d, dash, "\frac{\partial w}{\partial y}"] \arrow[dr, dash, "\frac{\partial w}{\partial z}"] & \\
x \arrow[dr, dash, "\frac{dx}{dt}"'] & y \arrow[d, dash, "\frac{dy}{dt}"] & z \arrow[dl, dash, "\frac{dz}{dt}"] \\
 & t &
\end{tikzcd}$$

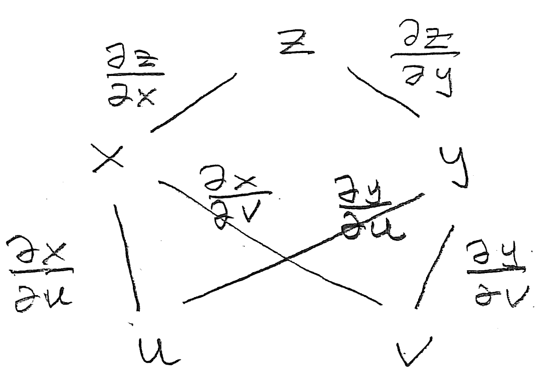
Label each edge with the corresponding derivative. To get $\frac{dw}{dt}$ look at all paths connecting w and t, multiply the derivatives along each path, and add contributions from each path.

<u>Case 2</u>: $z = f(x,y)$, $x = g(u,v)$, $y = h(u,v)$.

Then z depends on u, v:

$$z = f(x,y) = f(g(u,v), h(u,v)).$$

Find $\frac{\partial z}{\partial u}$ and $\frac{\partial z}{\partial v}$.



$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u}$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v}$$

Ex: $z = x^2 e^y$, $x = 4u$, $y = 3u^2 - 2v$.

Find $\left.\frac{\partial z}{\partial u}\right|_{(u,v)=(1,2)}$.

$$\frac{\partial z}{\partial u} = (2xe^y)\cdot 4 + x^2 e^y \cdot 6u$$

$(u,v) = (1,2) \longrightarrow x = 4$, $y = 3 - 4 = -1$

$$\left.\frac{\partial z}{\partial u}\right|_{(1,2)} = 8e^{-1}\cdot 4 + 16\cdot e^{-1}\cdot 6 = (32+96)e^{-1} = \frac{128}{e}$$

We will do an applied example later.