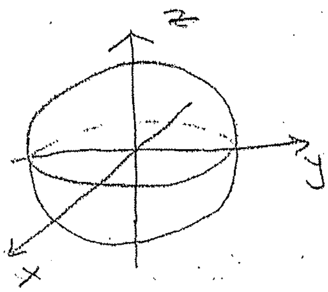
14.5 Gradients of Functions of Three Variables

Ex: $f(x,y,z) = x^2 + y^2 + z^2$

We cannot "graph" functions of three variables.
We can graph their level surface for a given k:

$$x^2 + y^2 + z^2 = k$$



We can, of course, define three partial derivatives at each point (a, b, c):

$$f_x(a,b,c),\ f_y(a,b,c),\ f_z(a,b,c)$$

by fixing two variables are differentiating with respect to the third. For example:

$$f_z(a,b,c) = \frac{d}{dz}\Big|_{z=c} [f(a,b,z)].$$

In our example:

$$f_x(x,y,z) = 2x,\ f_y(x,y,z) = 2y,\ f_z(x,y,z) = 2z.$$

We can define the directional derivative of a function of three variables $f(x,y,z)$ in the direction of $\vec{u} = u_1\vec{i} + u_2\vec{j} + u_3\vec{k}$, $\|\vec{u}\| = 1$:

$$f_{\vec{u}}(a,b,c) = \lim_{h \to 0} \frac{f(a+hu_1, b+hu_2, c+hu_3) - f(a,b,c)}{h}.$$

This is the rate of change in the direction of $\vec{u}$.

We can define the gradient vector;

$$\nabla f(x,y,z) = f_x(x,y,z)\vec{i} + f_y(x,y,z)\vec{j} + f_z(x,y,z)\vec{k}.$$

In our example, $f(x,y,z) = x^2 + y^2 + z^2$,

$$\nabla f = 2x\vec{i} + 2y\vec{j} + 2z\vec{k}.$$

As before, if $f(x,y,z)$ is differentiable at (a,b,c):

$$f_{\vec{u}}(a,b,c) = \nabla f(a,b,c) \cdot \vec{u} \qquad \frac{\text{units of } f}{\text{units of dist in dir } \vec{u}}$$

A before:

If $\nabla f(a,b,c) \neq \vec{0}$, then

- $\nabla f(a,b,c)$ points in the direction of the greatest rate of change of f at (a,b,c).
- $\|\nabla f(a,b,c)\|$ is this greatest rate of change
- $\nabla f(a,b,c)$ is perpendicular to the level surface through (a,b,c).

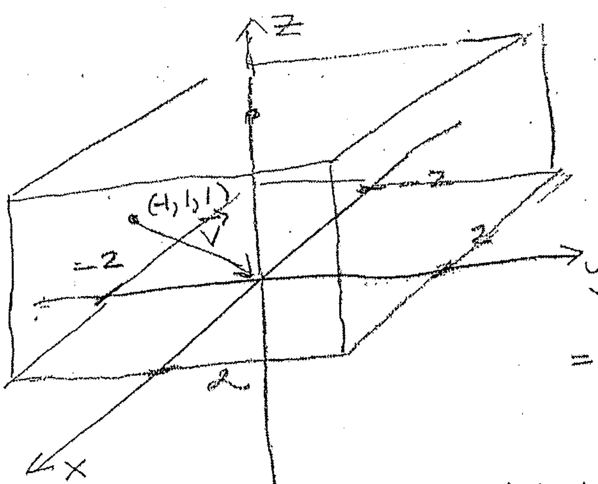
Ex Suppose that the function $F(x,y,z) = x^2 + y^4 + x^2z^2$ gives concentration of salt, in gr/gal, at any point (x,y,z) of a rectangular tank of water occupying the region

$$-2 \le x \le 2, \quad -2 \le y \le 2, \quad 0 \le z \le 2,$$

(All measurments in meters.) Suppose you are at the point $(-1,1,1)$.

(a) In what direction should you move if you want the concentration to increase the fastest?

(b) If you move from $(-1,1,1)$ toward the origin $(0,0,0)$, how fast is the concentration changing?



(a)

$\nabla F(x,y,z) =$

$= (2x + 2xz^2)\vec{i} + 4y^3\vec{j} + 2x^2z\vec{k}$

$\nabla F(-1,1,1) = -4\vec{i} + 4\vec{j} + 2\vec{k}$

The direction of greatest increase in concentration.

$$\|\nabla F(-1,1,1)\| = \sqrt{16+16+4} = 6\ \frac{g/gal}{m}$$

(b) $\vec{V} = \overrightarrow{(-1,1,1)(0,0,0)} = \vec{i} - \vec{j} - \vec{k}, \quad \|\vec{V}\| = \sqrt{3}$

$\vec{u} = \frac{1}{\sqrt{3}}\vec{i} - \frac{1}{\sqrt{3}}\vec{j} - \frac{1}{\sqrt{3}}\vec{k}$

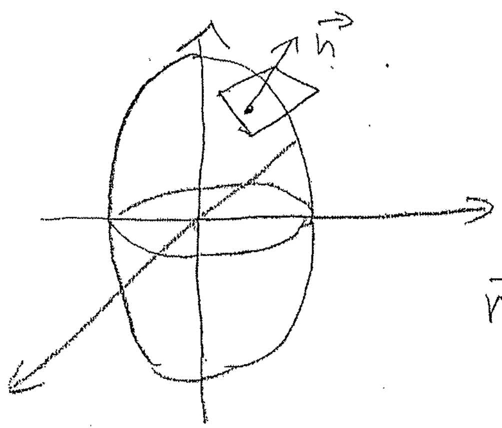
$F_{\vec{v}}(-1,1,1) = F_{\vec{u}}(-1,1,1) = \nabla F(-1,1,1) \cdot \vec{u} =$

$= -\frac{4}{\sqrt{3}} - \frac{4}{\sqrt{3}} - \frac{2}{\sqrt{3}} = -\frac{10}{\sqrt{3}} \cong -5.77 \ \frac{g/gal}{m}.$

Ex: Find the equation of the tangent plane to the ellipsoid $x^2 + 2y^2 + z^2 = 15$ at $(2, 1, 3)$.

The ellipsoid is the level surface of

$F(x, y, z) = x^2 + 2y^2 + z^2$



$\vec{n} = \nabla F(2,1,3)$

$\nabla F = 2x\vec{i} + 4y\vec{j} + 2z\vec{k}$

$\vec{n} = \nabla F(2,1,3) = 4\vec{i} + 4\vec{j} + 6\vec{k}$

$P = (2, 1, 3)$

Plane: $4(x-2) + 4(y-1) + 6(z-3) = 0.$