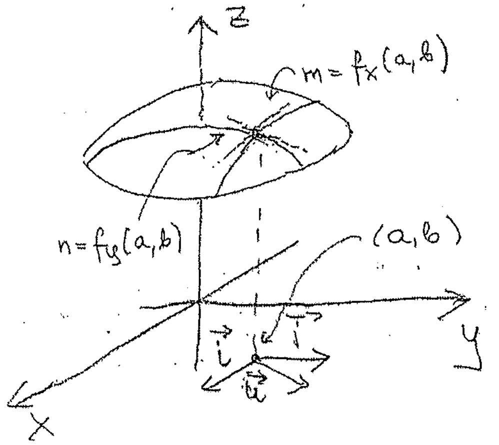
14.4 Gradients. Directional Derivatives

Let $z = f(x,y)$ be given. The partial derivatives $f_x(a,b)$, $f_y(a,b)$ at some point (a,b) give the rates of change in the direction of x and the diretion of y:



$f_x(a,b)$ gives us the rate of change when we walk from (a,b) in the direction of $\vec{i}$, $f_y(a,b)$ if we walk in the direction of $\vec{j}$.

What is the rate of change if we move from (a,b) in the direction of some other unit vector $\vec{u}$?

This rate of change is called the directional derivative of $f(x,y)$ at (a,b) in the direction of $\vec{u}$.

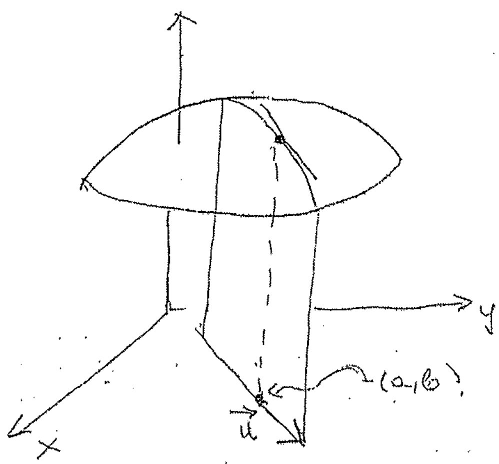
<u>Def</u>: Let $\vec{u} = u_1\vec{i} + u_2\vec{j}$, $\|\vec{u}\| = 1$, $z = f(x,y)$, (a,b) be given. The directional derivative of $f(x,y)$ at (a,b) in the direction of $\vec{u}$ is:

$$f_{\vec{u}}(a,b) = \lim_{h \to 0} \frac{f(a+hu_1,\, b+hu_2) - f(a,b)}{h}$$

$$(a,b) \longrightarrow (a+hu_1,\, b+hu_2).$$

Note: The definition talks about both-sided limit "$\lim_{h \to 0}$". Thus h can be positive or negative.

The limit exists if the cross-section of the graph by the vertical plane containing $\vec{u}$ and $(a, b, f(a,b))$ has the tangent line at $(a, b, f(a,b))$:



Thus $f_{\vec{u}}(a,b)$ also tells us that if we go in the opposite direction, $-\vec{u}$, $f(x,y)$ will be changing at the rate $-f_{\vec{u}}(a,b) = f_{-\vec{u}}(a,b)$

Indeed:

$$f_{-\vec{u}}(a,b) = \lim_{h \to 0} \frac{f(a+h(-u_1), b+h(-u_2)) - f(a,b)}{h} =$$

$$= \lim_{h \to 0} \frac{f(a+(-h)u_1, b+(-h)u_2) - f(a,b)}{-h} =$$

$$= -f_{\vec{u}}(a,b).$$

Note, the displacement vector from (a, b) to $(a+hu_1, b+hu_2)$ is $\vec{d} = (h\,u_1)\,\vec{i} + (h\,u_2)\,\vec{j} = h \cdot \vec{u}$. So $\|\vec{d}\| = h$ (or more precisely $|h|$).

This is why we need $\vec{u}$ to be the unit vector so the magnitude of h is the magnitude of the displacement. Geometrically, $f_{\vec{u}}(a, b)$ is the slope of the cross-section with the vertical plane parallel to $\vec{u}$ through $(a, b, f(a, b))$. Clearly:

$$f_x(a, b) = f_{\vec{i}}(a, b), \quad f_y(a, b) = f_{\vec{j}}(a, b).$$

<u>Def</u>: Let $f(x, y)$, (a, b), and a vector $\vec{v}$ be given. Then

$$f_{\vec{v}}(a, b) = f_{\frac{\vec{v}}{\|\vec{v}\|}}(a, b).$$

How do we calculate directional derivatives? Using the so-called gradient vector.

<u>Def</u>: The gradient vector of $f(x, y)$ at (a, b) is defined as

$$\operatorname{grad} f(a, b) = f_x(a, b)\,\vec{i} + f_y(a, b)\,\vec{j}.$$

In general:

$$\text{grad}\, f(x,y) = f_x(x,y)\vec{i} + f_y(x,y)\vec{j}$$

or

$$\text{grad}\, f = f_x\vec{i} + f_y\vec{j}$$

Ex: Let $f(x,y) = x^3y + 3y + x$. Find $\text{grad}\, f(x,y)$. Find $\text{grad}\, f(1,0)$.

$$f_x = 3x^2y + 1, \quad f_y = x^3 + 3$$

$$\text{grad}\, f = (3x^2y+1)\vec{i} + (x^3+3)\vec{j}$$

$$\underline{\text{grad}\, f(1,0) = \vec{i} + 4\vec{j}}$$

grad f is a vector!

Th: Let $f(x,y)$ be differentiable at (a,b), $\vec{u}$ be the unit vector. Then

$$f_{\vec{u}}(a,b) = \text{grad}\, f(a,b)\cdot\vec{u} =$$

$$= f_x(a,b)\,u_1 + f_y(a,b)\,u_2.$$

Gradient is also denoted:

$$\underline{\text{grad}\, f = \nabla f}$$

Ex: Let $\vec{u} = \frac{1}{\sqrt{2}}\vec{i} + \frac{1}{\sqrt{2}}\vec{j}$, $f(x,y) = x^2+y^2$.

Find $f_{\vec{u}}(1,0)$.

We use gradient.

$$\operatorname{grad} f(x,y) = 2x\vec{i} + 2y\vec{j}$$

$$\operatorname{grad} f(1,0) = 2\vec{i}$$

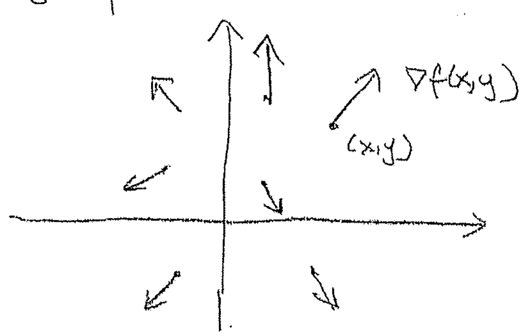
$$f_{\vec{u}}(1,0) = 2\vec{i}\cdot\vec{u} = 2\cdot\frac{1}{\sqrt{2}} = \sqrt{2}.$$

So finding directional derivatives is as easy as finding gradients.

Let $f(x,y)$, differentiable, be given. Then we have the gradient vector at each point:

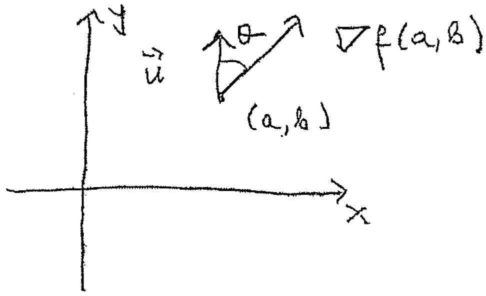
$$\nabla f(x,y)$$

So we have a vector field – the gradient field on the xy-plane:



Gradient Geometrically

Let's have $f(x,y)$, (a,b), $\nabla f(a,b)$:



How about directional derivatives at (a,b)?

Let $\vec{u}$ be a unit vector.

$$f_{\vec{u}}(a,b) = \nabla f(a,b)\cdot\vec{u} = \|\nabla f(a,b)\|\cdot\underbrace{\|\vec{u}\|}_{1}\cos(\theta)$$

$f_{\vec{u}}(a,b)$ is the largest if $\cos(\theta)=1$; that is, when $\theta=0$, so $\vec{u}$ is parallel to $\nabla f(a,b)$.

Thus:

Th: Let $f(x,y)$ be differentiable at (a,b), $\nabla f(a,b)\neq\vec{0}$.

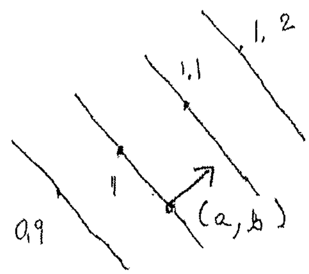
Then:

(1) $\nabla f(a,b)$ points in the direction of the largest rate of change of f at (a,b). (The direction of the fastest growth.)

(2) This largest rate of change is $\|\nabla f(a,b)\|$.

(3) $\nabla f(a,b)$ is perpendicular to the contour line of $f(x,y)$ through (a,b).

Why (3)? Since $f(x,y)$ is differentiable at (a,b), locally near (a,b) the graph of $z=f(x,y)$ is flat. Thus, locally contour lines are parallel:

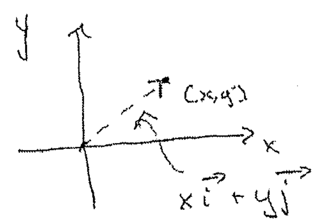


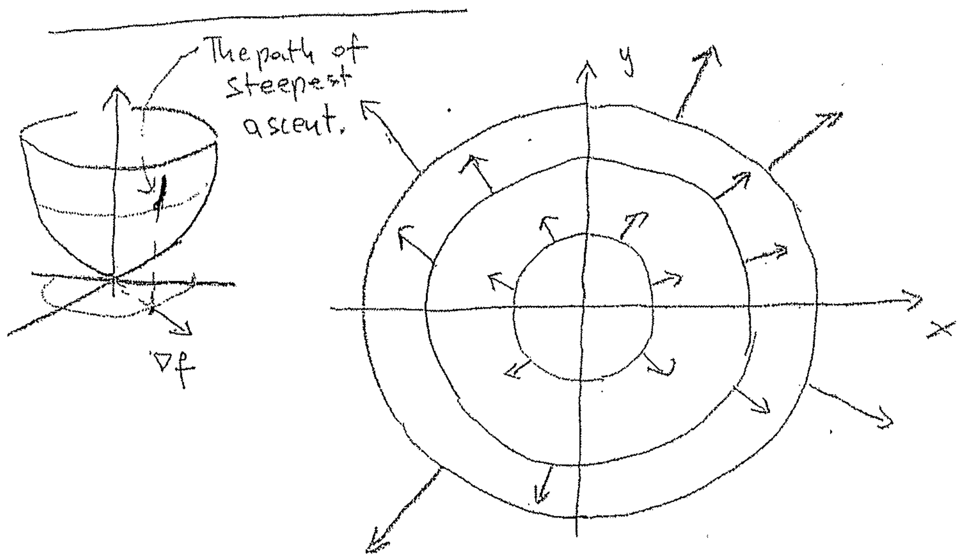
You obtain fastest growth by moving perpendicularly to contours in the direction of increasing values. So $\nabla f(a,b)$ that points toward fastest increase must be perpedicular to contours.

Ex: Let $f(x,y) = x^2+y^2$. Sketch the contour diagram and the gradient field in one coordinate system.

$$\nabla f(x,y) = 2x\vec{i} + 2y\vec{j}$$

parallel the vector from $(0,0)$ to (x,y).





Magnitudes of vectors are scaled.

Ex A square metal plate is placed in the xy-plane in such a way that $0 \le x \le 3$, $0 \le y \le 3$ (All measurements in meters.). The temperature at each point (x,y) of the plane is given by:

$$T(x,y) = \frac{100}{x^2+y^2+1} \quad \text{in } {}^\circ F.$$

(a) Find the direction of the greatest increase in temp. at $(1,2)$. What is the greatest rate of increase at the point $(1,2)$?

(b) Find the rate of increase at $(1,2)$ in the direction $\vec{u} = \frac{2}{\sqrt{5}}\vec{i} + \frac{1}{\sqrt{5}}\vec{j}$.

Clearly, we have to find the gradient $\nabla T(x,y)$ first.

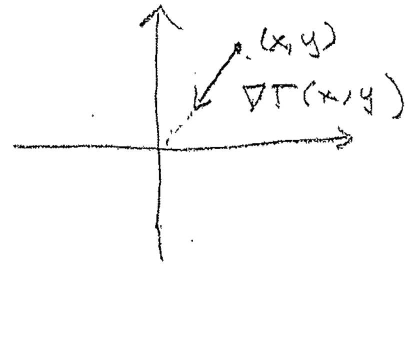
$$T_x(x,y) = \frac{\partial}{\partial x}\left[\frac{100}{x^2+y^2+1}\right] = \frac{\partial}{\partial x}\left[100(x^2+\underbrace{y^2+1}_{\text{Const}})^{-1}\right] =$$

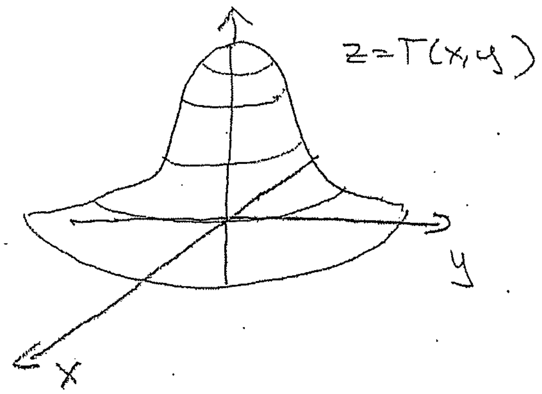
$$= 100 \cdot -(x^2+y^2+1)^{-2} \cdot 2x =$$

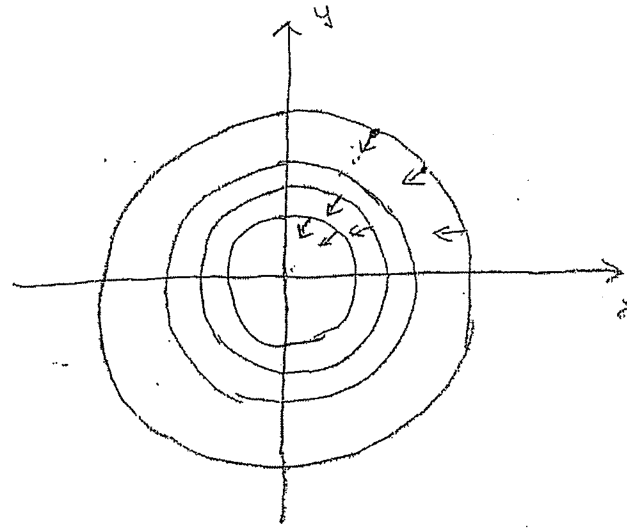
$$= -\frac{100}{(x^2+y^2+1)^2} \cdot 2x = T_x(x,y)$$

$$T_y(x,y) = -\frac{100}{(x^2+y^2+1)^2} \cdot 2y$$

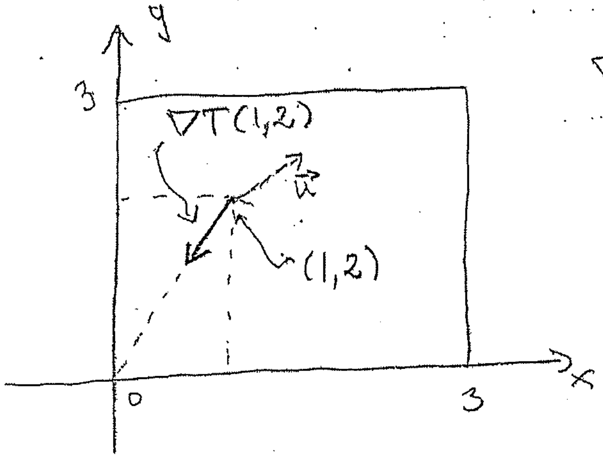
$$\nabla T(x,y) = -\frac{100}{(x^2+y^2+1)^2}\left(2x\vec{i} + 2y\vec{j}\right)$$







Contours are concentric circles centered at (0,0).
At each point the gradient $\nabla T(x,y)$ points toward the origin.



$$\nabla T(1,2) = -\frac{200}{36}\vec{i} - \frac{400}{36}\vec{j}$$

The direction of the greatest increase in temprature at (1,2). (Toward the origin). This answers (a).

(b) $\|\nabla T(1,2)\| = \sqrt{\left(\frac{200}{36}\right)^2 + \left(\frac{400}{36}\right)^2} \cong 12.42 \ \frac{°F}{m}$

(c) $T_{\vec{u}}(1,2) = \nabla T(1,2) \cdot \vec{u} =$

$$= -\frac{200}{36} \cdot \frac{2}{\sqrt{5}} - \frac{400}{36} \cdot \frac{1}{\sqrt{5}} \cong -9.93 \ \frac{°F}{m}$$