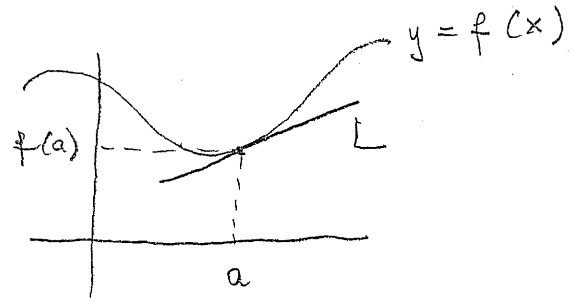
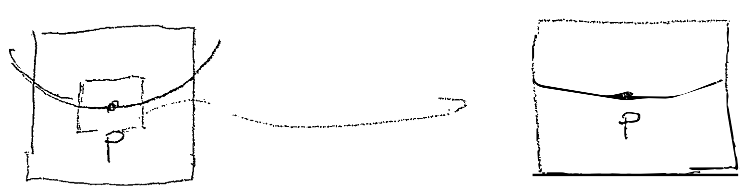
(14.3) Differentiability, Tangent Plane, Local Linearization

A function of one variable, $y = f(x)$, is called <u>differentiable</u> at $x = a$ if $f'(a)$ exists which is when the graph $y = f(x)$ has the <u>tangent line</u> at $(a, f(a))$:

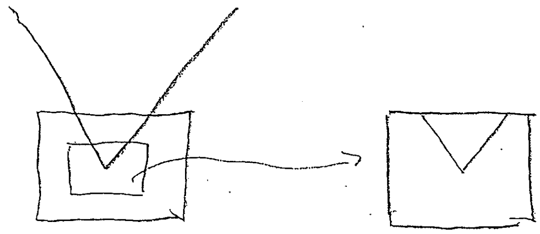


$$f'(a) = m_{tan}^{(a, f(a))}$$

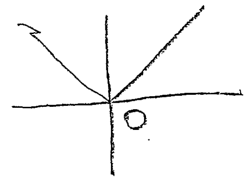
The tangent line exists if the graph of $y = f(x)$ "flattens" to a straight line when we look at smaller and smaller portions of the graph around the point $P = (a, f(a))$:



Not every curve around each point "flattens" to a straight line:



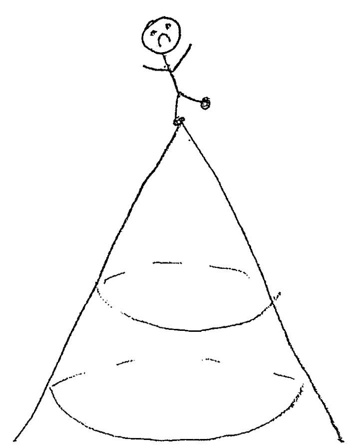
<u>Ex</u>: $f(x) = |x|$ at $a = 0$:



No tangent line.

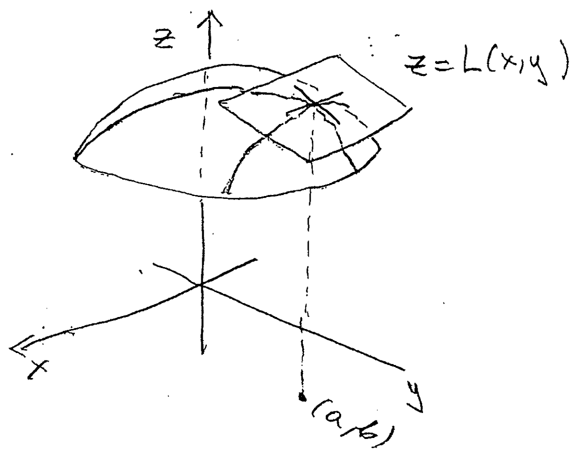
Curves that "flatten" to straight lines as we look at smaller and smaller portions are called "smooth".

How about surfaces in 3D and graphs of functions $z = f(x, y)$? A "nice", "smooth" surface "flattens" to a plane when we look at small portions of it:



That plane it calls the tangent plane.

Let $z = f(x,y)$, $(x,y) = (a,b)$ be given.
$f(x,y)$ is called <u>differentiable</u> at (a,b) if
$z = f(x,y)$ <u>has a tangent plane</u> at $(a,b,f(a,b))$:



Let's denote by $L(x,y)$ the linear function whose graph

$$z = L(x,y)$$

is the tangent plane.
What is the formula for $L(x,y)$; that is, what is the equation of the tangent plane?
The tangent lines to the cross-sections $f(x,b)$ and $f(a,y)$ have to be on the tangent plane. Those tangent lines have slopes $f_x(a,b)$ and $f_y(a,b)$. Thus, the slope of $z = L(x,y)$ in the x-direction is $f_x(a,b)$, the slope in the y direction is $f_y(a,b)$. The plane passes through $(a,b,f(a,b))$.
Thus the equation of the tanget plane at (a,b) is:

$$z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b).$$

The linear function whose graph is the tangent plane:

$$L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b).$$

Clearly:

$$f(x,y) \cong L(x,y) \quad \text{for } (x,y) \text{ close to } (a,b).$$

$L(x,y)$ is called the local linearization of $f(x,y)$ at (a,b).

Ex: Find the equation of the tangent plane (assuming it exists) to

$$z = f(x,y), \quad f(x,y) = y e^{\frac{x}{y}} \text{ at } (1,1,e).$$

$$z = e + f_x(1,1)(x-1) + f_y(1,1)(y-1)$$

$$f_x = \frac{\partial}{\partial x}\left[y e^{\frac{x}{y}}\right] = e^{\frac{x}{y}}, \quad f_x(1,1) = e$$

$$f_y = \frac{\partial}{\partial y}\left[y e^{\frac{x}{y}}\right] = e^{\frac{x}{y}} + y\cdot\left(-\frac{x}{y^2}\right)e^{\frac{x}{y}} = e^{\frac{x}{y}} - \frac{x}{y}e^{\frac{x}{y}}$$

$$f_y(1,1) = e - e = 0$$

L: $z = e + e(x-1) = ex$ $\quad \boxed{z = ex}$

Since

$$f(x,y) \cong L(x,y) \quad \text{for } (x,y) \text{ close to } (a,b),$$

we have the following approximation formula:

$$\underline{f(x,y) \cong f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)}$$

for $(x-a)$, $(y-b)$ "small".

In other words:

$$f(a+\Delta x, b+\Delta y) \cong f(a,b) + f_x(a,b)\Delta x + f_y(a,b)\Delta y,$$

$$\Delta x = x-a, \quad \Delta y = y-b.$$

<u>Ex</u>: An unevenly heated plate has temperature $T(x,y)$ in °C at the point (x,y). If

$$T(2,1) = 135, \quad T_x(2,1) = 16, \quad T_y(2,1) = -15,$$

estimate the temperature at $(2.04, 0.97)$.

$$T(2.04, 0.97) \cong T(2,1) + T_x(2,1)\cdot\Delta x + T_y(2,1)\cdot\Delta y =$$

$\Delta x = 0.04$ $\quad$ $\Delta y = -0.03$

$$\doteq 135 + 16 \cdot 0.04 + (-15)\cdot(-0.03) = \underline{136.09\ °C}$$

The Differential

Let $z = f(x,y)$. We have:

$$f(x,y) \cong f(a,b) + f_x(a,b) \cdot \Delta x + f_y(a,b) \cdot \Delta y$$

Denote

$$\Delta f = f(x,y) - f(a,b).$$

Then:

$$\Delta z = \Delta f \cong f_x(a,b) \cdot \Delta x + f_y(a,b) \cdot \Delta y$$

The infinitesimal version of this formula is called the differential, df or dz, of $f(x,y)$ at (a,b):

$$df = f_x(a,b)\,dx + f_y(a,b)\,dy$$

In general:

$$df = f_x\,dx + f_y\,dy$$

Ex: Compute the differential of

$$z = x \sin(xy).$$

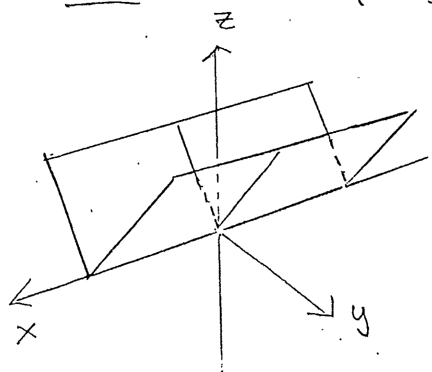
$$\frac{\partial z}{\partial x} = \sin(xy) + xy\cos(xy), \quad \frac{\partial z}{\partial y} = x^2\cos(xy)$$

Thus:

$$dz = (\sin(xy) + xy\cos(xy))\,dx + (x^2\cos(xy))\,dy$$

When is a given function $z = f(x,y)$ differentiable at (a,b)? Is it enough that $f_x(a,b)$, $f_y(a,b)$ exist? Not really. Let's look at a couple of examples.

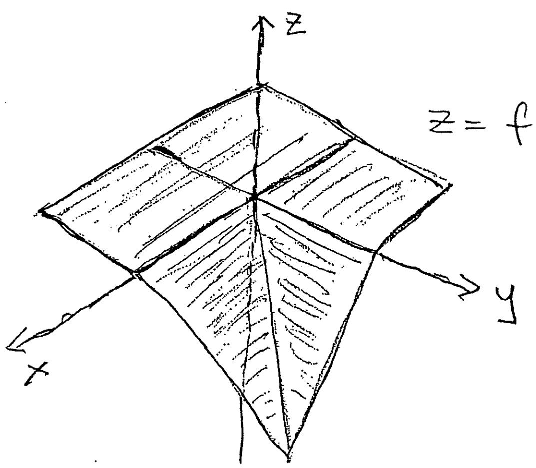
Ex $z = f(x,y)$, $f(x,y) = |y|$.



$f_x(0,0) = ?$ $f_y(0,0) = ?$

Look at cross-sections: $f(x,0)$, $f(0,y)$.

Clearly, $f_x(0,0) = 0$, $f_y(0,0)$ does not exist. No tangent plane at $(0,0)$ (or any point $(a,0)$). It can happen, however, that $f_x(a,b)$, $f_y(a,b)$ both exist and there is no tangent plane:



$z = f(x,y)$. What about $f_x(0,0)$, $f_y(0,0)$? What about differentiability of $f(x,y)$ at $(0,0)$?

Differentiability for $f(x,y)$ is a bit complicated.