14.1, 14.2 Cont'd

Let $z = f(x,y)$. We defined partial derivatives functions denoted:

$$f_x(x,y) = \frac{\partial z}{\partial x} = \frac{\partial}{\partial x}[f(x,y)] = z_x = \frac{\partial f}{\partial x} = f_x$$

$$f_y(x,y) = \frac{\partial z}{\partial y} = \frac{\partial}{\partial y}[f(x,y)] = z_y = \frac{\partial f}{\partial y} = f_y$$

Computing partials algebraically is easy: we consider one variable to be a constant and differentiate with respect to the other.

Ex: Let $f(x,y) = x^3 + 3x^2y + y^2$. Find $f_x(x,y)$ and $f_y(x,y)$.

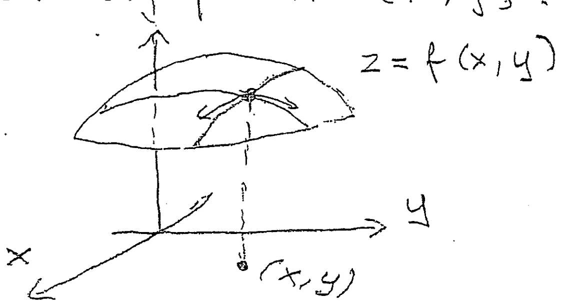
$$f_x(x,y) = \frac{\partial}{\partial x}[x^3 + 3x^2y + y^2] = 3x^2 + 6xy$$

y constant,
take the derivative in x.

$$f_y(x,y) = \frac{\partial}{\partial y}[x^3 + 3x^2y + y^2] = 3x^2 + 2y$$

x constant,
take the derivative in y.

Geometrically, we found slopes of $z = f(x,y)$ in the x and y directions at any point (x,y):



Ex : $f(t, s) = t^2 e^{t \cdot s}$

$$\frac{\partial f}{\partial t} = \frac{\partial}{\partial t}\left[t^2 e^{t \cdot s} \right] = 2t e^{ts} + t^2 s e^{ts}$$

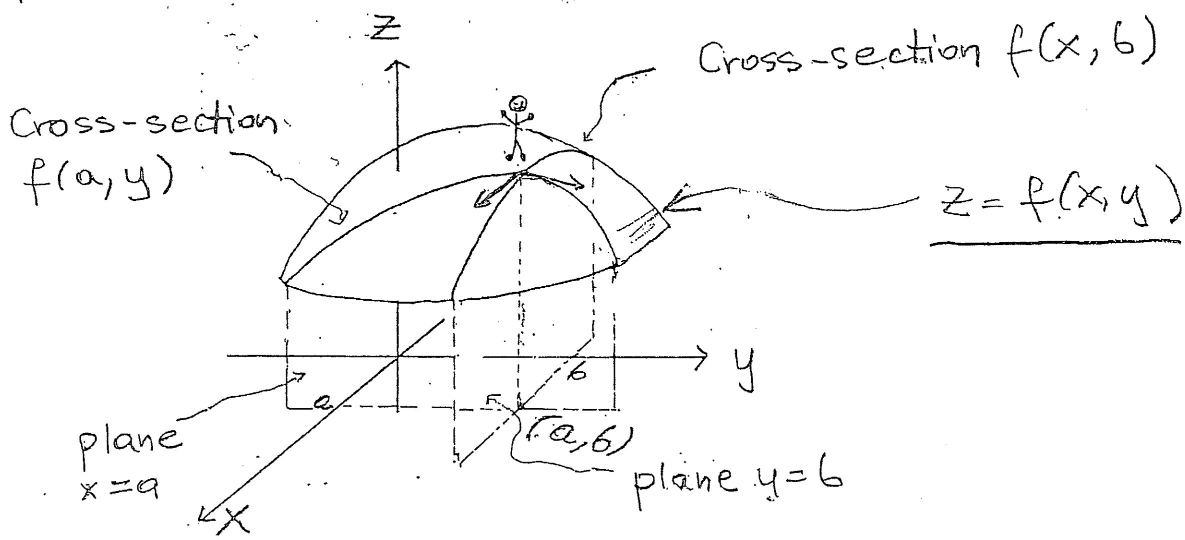
↑ s constant

$$\frac{\partial f}{\partial s} = \frac{\partial}{\partial s}\left[t^2 e^{ts} \right] = t^2 . t e^{ts} = t^3 e^{ts}$$

↑ t constant

Practice !

Since partial derivatives are ordinary derivatives of cross-sections, they can be interpreted as rates of change.



$$f_x(a, b) = \frac{d}{dx}\Big|_{x=a}\left[f(x, b) \right] = \lim_{\Delta x \to 0} \frac{f(a + \Delta x, b) - f(a, b)}{\Delta x}$$

$f_x(a, b)$ $\dfrac{\text{units of } z}{\text{unit of } x}$ - rate of change of z with respect to x at $x = a$ with y fixed at $y = b$.

since

$$f_x(a,b) = \lim_{\Delta x \to 0} \frac{f(a+\Delta x, b) - f(a,b)}{\Delta x}$$

we can approximate:

$$(*) \quad f_x(a,b) \cong \frac{f(a+\Delta x, b) - f(a,b)}{\Delta x} \quad \text{for } \Delta x \text{ "small".}$$

Often used if we have a contour diagram or a table of values for $f(x,y)$.

All goes the same for $f_y(a,b)$:

$$f_y(a,b) = \frac{d}{dy}\Big|_{y=b} [f(a,y)] = \lim_{\Delta y \to 0} \frac{f(a, b+\Delta y) - f(a,b)}{\Delta y}$$

$f_y(a,b)$ $\frac{\text{units of } z}{\text{unit of } y}$ - rate of change of z with respect to y at $y=b$ with $x=a$ fixed.

$$(*) \quad f_y(a,b) \cong \frac{f(a, b+\Delta y) - f(a,b)}{\Delta y} \quad \text{for } \Delta y \text{ "small".}$$

The approximation formulas $(*)$ give:

$$f(a+\Delta x, b) \cong f(a,b) + f_x(a,b) \cdot \Delta x \quad \text{for } \Delta x \text{ "small"}$$

$$f(a, b+\Delta y) \cong f(a,b) + f_y(a,b) \cdot \Delta y \quad \text{for } \Delta y \text{ "small"}$$

All as in MTH 141 as one variable is fixed.

Ex. Suppose that your weight, w, in pounds, is a function $w = f(c, m)$ of the number of calories, c, you consume daily and the number of minutes, m, you exercise daily.

(a) What are the units of $\frac{\partial w}{\partial c}(c, m)$?

(b) What is the practical meaning of the statement $\frac{\partial w}{\partial c}(2100, 20) = 0.007$?

(c) What are the units of $\frac{\partial w}{\partial m}(c, m)$?

(d) What is the practical meaning of the statement $\frac{\partial w}{\partial m}(2100, 20) = -0.2$?

(e) Assume that $w(2100, 20) = 120$, $\frac{\partial w}{\partial m}(2100, 20) = -0.2$. Estimate $w(2100, 25)$.

$$\underset{\text{lb}}{w} = f(\underset{\text{cal}}{c}, \underset{\text{min}}{m})$$

(a) $\frac{\partial w}{\partial c}(c, m)$ $\quad \frac{\text{lb}}{\text{cal}}$

(With minutes of excercise fixed, how fast is your weight increasing when number of calories increases.)

(b) $\frac{\partial w}{\partial c}(2100, 20) = 0.007 \frac{\text{lb}}{\text{cal}}$

When you excercise 20 min daily and consume 2100 calories daily, your weight is increasing at the (instantaneous) rate of $0.007 \frac{\text{lb}}{\text{cal}}$ if you increase your calories (and keep 20 min constant).

(c) $\frac{\partial w}{\partial m}(c, m)$ $\quad \frac{\text{lb}}{\text{min}}$

(With calories fixed how fast is your weight decreasing as you increase minutes of excercise.)

(d) $\frac{\partial w}{\partial m}(2100, 20) = -0.2 \frac{lb}{min}$

When you excercise 20 min daily and cousume 2100 calories your weight is changing at the (instantaneous) rate of $-0.2 \frac{lb}{min}$ as you increase minutes of excercise. In other words, for each additional minute you will loose approximately 0.2 lb.

(e) $\underset{cal}{w(2100}, \underset{min}{20)} = \underset{lb}{120}$, $\frac{\partial w}{\partial m}(2100, 20) = -0.2 \frac{lb}{min}$

Given that, estimate:

$w(2100, 25)$

$$w(2100, 25) \cong 120\,lb + (-0.2)\frac{lb}{min} \cdot 5\,min = 119\,lb$$

Approximately as $\frac{\partial w}{\partial m}(2100, 20) = -0.2 \frac{lb}{min}$ is the instantaneous rate at $c = 2100$ and $m = 20$ and it may not stay coutant between (2100, 20) and (2100, 25).

(f) Estimate $w(2200, 20)$.

$$w(\underset{2100+\underbrace{100}_{\Delta c}}{2200}, 20) \cong w(2100, 20) + \frac{\partial w}{\partial c}(2100, 20)\Delta c = 120\,lb + \left(0.007\frac{lb}{cal}\right)100\,cal = 120.7\,lb$$

What if we want to change both c and m:

$w(2100 + \Delta c, 20 + \Delta m) \cong$?

Ex

For Problems (a), (b) refer to Table 9.5 giving the wind-chill factor, C in °F, as a function $f(w,T)$ of the wind speed, w, and the temperature, T. The wind-chill factor is a temperature which tells you how cold it feels, as a result of the combination of wind and temperature.

TABLE 9.5 *Wind-chill factor (°F)* ← T →

↑ W ↓

mph\°F	35	30	25	20	15	10	5	0
5	33	27	21	16	12	7	0	−5
10	22	16	10	3	−3	−9	−15	−22
15	16	9	2	−5	−11	−18	−25	−31
20	12	4	−3	−10	−17	−24	−31	−39
25	8	1	−7	−15	−22	−29	−36	−44

↑ $f(w,25)$

(a) Estimate $f_w(10,25)$. What does your answer mean in practical terms?

(b) Estimate $f_T(5,20)$. What does your answer mean in practical terms?

$$C = f(w, T)$$

wind-chill in °F

wind speed in mph

temperature outside in °F

(a) $f_w(10,25)$

T stays constant at 25°F, wind speed changes.

The best estimate:

Increasing w: $\dfrac{2\,°F - 10\,°F}{5\text{ mph}} = -\dfrac{8}{5}\,\dfrac{°F}{\text{mph}}$

Decreasing w: $\dfrac{10\,°F - 21\,°F}{5\text{ mph}} = -\dfrac{11}{5}\,\dfrac{°F}{\text{mph}}$

$$f_w(10,25) \cong -\frac{8}{5}\,\frac{°F}{\text{mph}} = -1.6\,\frac{°F}{\text{mph}}$$

$$\cong -\frac{11}{5}\,\frac{°F}{\text{mph}} = -2.2\,\frac{°F}{\text{mph}}$$

Or average them:

$$f_w(10,25) \cong \frac{-3.8}{2} = 1.9\,\frac{°F}{\text{mph}}.$$