

14.1, 14.2 Partial Derivatives

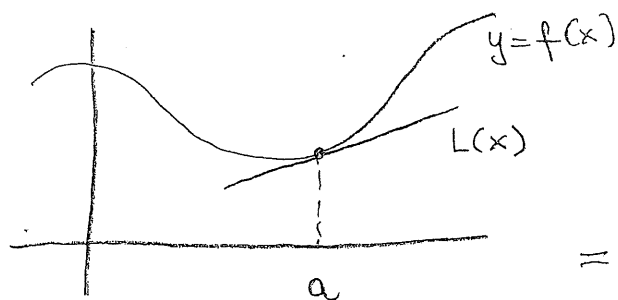
Going back to functions of two variables:

$$z = f(x, y).$$

Today we are going to define and interpret partial derivatives of  $f(x, y)$  with respect to  $x$  and  $y$ .

Recall for a function of one variable:

$$y = f(x)$$



$$f'(a) = \left. \frac{dy}{dx} \right|_{x=a} = m_{\text{tan}}$$

$$= \left( \begin{array}{l} \text{The rate of change} \\ \text{of } y \text{ at } x=a \end{array} \right) \frac{\text{units of } y}{\text{unit of } x}$$

$$f'(a) = \lim_{h \rightarrow 0} \frac{f(a+h) - f(a)}{h}$$

So  $f(x) \approx f(a) + f'(a)(x-a) \leftarrow (x = a+h)$

The equation of the tangent line:

$$L(x) = f(a) + f'(a)(x-a)$$

$L(x)$  local linearization of  $f(x)$  at  $x=a$ .

$f(x)$  differentiable at  $x=a \iff$  the tangent line exists.

How much of that translates to  $f(x, y)$  and how?

The first step: partial derivatives.

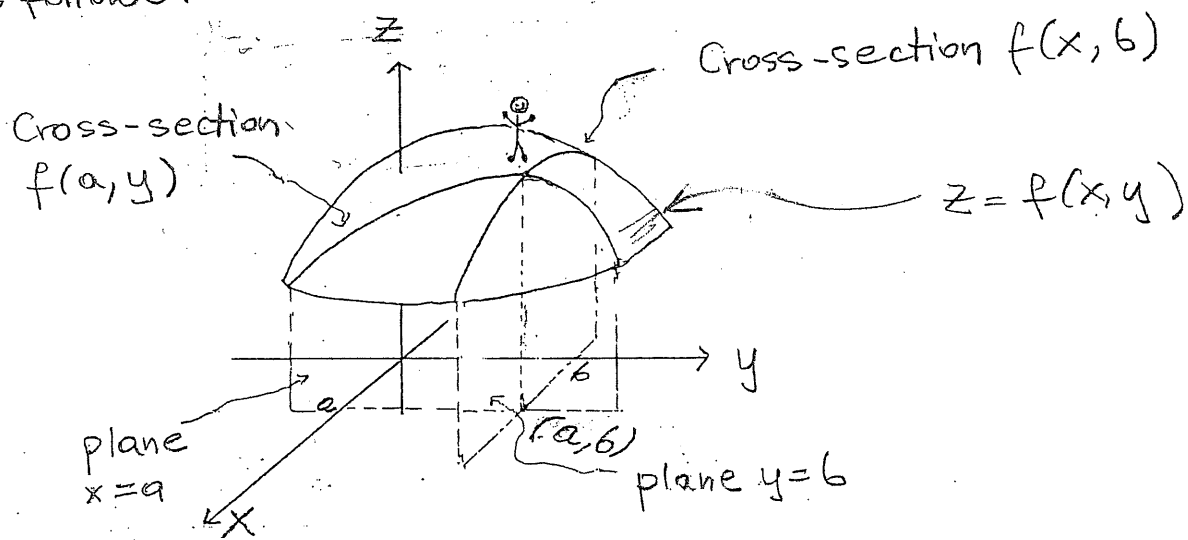
Let a function  $z = f(x, y)$  and a point  $(a, b)$  in its domain be given. We define:

$f_x(a, b)$  - the partial derivative of  $f(x, y)$  with respect to  $x$  at  $(a, b)$

and

$f_y(a, b)$  - the partial derivative of  $f(x, y)$  with respect to  $y$  at  $(a, b)$ .

as follows.



$$f_x(a, b) = \frac{d}{dx} \Big|_{x=a} [ f(x, b) ]$$

$$f_y(a, b) = \frac{d}{dy} \Big|_{y=b} [ f(a, y) ]$$

In other words

$f_x(a, b)$  = the slope of  $f(x, b)$  at  $x=a$ .

$f_y(a, b)$  = the slope of  $f(a, y)$  at  $y=b$ .

So to calculate

$$f_x(a, b)$$

we fix  $y = b$ , consider the function  $f(x, b)$  which is a function of one variable  $x$ , and calculate the ordinary derivative of  $f(x, b)$  with respect to  $x$ .

Similarly, to calculate

$$f_y(a, b)$$

we fix  $x = a$ , consider  $f(a, y)$  etc.

Ex: Let  $f(x, y) = x^2 + y^3$ . Find

$$f_x(2, 1), f_x(1, 3).$$

To find  $f_x(2, 1)$ , we fix  $y = 1$  and consider the cross-section

$$f(x, 1) = x^2 + 1$$

Now we take the ordinary derivative

$$\frac{d}{dx} [f(x, 1)] = \frac{d}{dx} [x^2 + 1] = 2x.$$

And evaluate at  $x = 2$ :  $f_x(2, 1) = 2x|_{x=2} = 4$

For  $f_x(1, 3)$ , we take

$$f(x, 3) = x^2 + 27$$

$$\frac{d}{dx} |_{x=1} [x^2 + 27] = 2x|_{x=1} = 2.$$

Clearly calculating  $f_x(a, b)$ ,  $f_y(a, b)$  for each given point separately is silly. For a given

$$f(x, y)$$

we should calculate partial derivatives functions:

$$f_x(x, y), f_y(x, y)$$

and then evaluate at any point  $(a, b)$  we want. How to do this?

Ex: Let  $f(x, y) = x^2 + y^3$ . Find

$f_x(x, y)$  and  $f_y(x, y)$ . Find  $f_y(2, 1)$ .

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To find  $f_x(x, y)$ , we take  $f(x, y) = x^2 + y^3$  and assume that  $y$  is fixed so  $y$  is a constant. Assuming that  $y$  is a constant, we take the derivative of  $f(x, y) = x^2 + y^3$  with respect to  $x$ :

$$f_x(x, y) = (x^2 + y^3)_x = 2x$$

To find  $f_y(x, y)$ , we assume that  $x$  is a constant and differentiate with respect to  $y$ :

$$f_y(x, y) = (x^2 + y^3)_y = 3y^2.$$

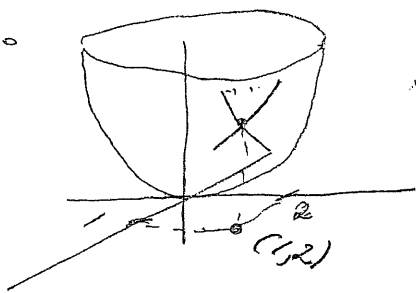
So  $f_y(2, 1) = 3$ . Easy!

Ex: Let  $f(x, y) = x^2 + y^2$ . Find  $f_x(1, 2)$ ,  $f_y(1, 2)$ .

$$f_x(x, y) = 2x, \quad f_y(x, y) = 2y$$

$$f_x(1, 2) = 2, \quad f_y(1, 2) = 4.$$

5.



For the paraboloid at  $(1, 2)$  the slope in the  $x$  direction is 2, the slope in the  $y$  direction is 4.

Leibnitz Notation:

$$z = f(x, y), \quad f_x(x, y) = \frac{\partial z}{\partial x} = \frac{\partial}{\partial x} [f(x, y)],$$

$$f_y(x, y) = \frac{\partial z}{\partial y} = \frac{\partial}{\partial y} [f(x, y)]$$

$$f_x(a, b) = \frac{\partial z}{\partial x} \Big|_{(a, b)}, \quad f_y(a, b) = \frac{\partial z}{\partial y} \Big|_{(a, b)}$$

Ex:  $z = x^2 y$ . Find  $\frac{\partial z}{\partial x}$ ,  $\frac{\partial z}{\partial y}$ .

$$\frac{\partial z}{\partial x} = \frac{\partial}{\partial x} [x^2 y] = 2yx, \quad \frac{\partial z}{\partial y} = \frac{\partial}{\partial y} [x^2 y] = x^2.$$

For a given  $z = f(x, y)$ , find partial derivatives  $\frac{\partial z}{\partial x}$ ,  $\frac{\partial z}{\partial y}$ .

Ex:  $z = \sin(xy^3)$ .

$$\frac{\partial z}{\partial x} = \cos(xy^3) \cdot y^3 \quad (\sin(x \cdot 5))'_x$$

$$\frac{\partial z}{\partial y} = \cos(xy^3) \cdot 3xy^2 \quad (\sin(5y^3))'_x$$

Ex:  $z = x^2 e^{xy}$

$$\frac{\partial z}{\partial x} = 2x e^{xy} + x^2 y e^{xy}$$

$$\frac{\partial z}{\partial y} = x^2 \cdot x e^{xy} = x^3 e^{xy}$$

Practice!

Of course partial derivatives,  $f_x(a, b)$ ,  $f_y(a, b)$  are rates of change of  $f(x, y)$  at  $(a, b)$  in the  $x$  direction and the  $y$  direction.