13.4 Cross Product

Let $\vec{v} = v_1\vec{i} + v_2\vec{j} + v_3\vec{k}$, $\vec{w} = w_1\vec{i} + w_2\vec{j} + w_3\vec{k}$

be given vectors.

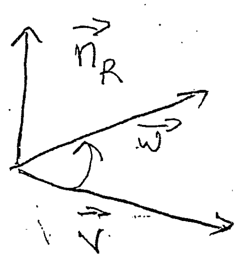
$\vec{v} \cdot \vec{w}$ is a number.

$\vec{v} \times \vec{w}$ is a vector.

$\vec{v} \times \vec{w}$ can be defined geometrically or algebraically.

A few remarks first.

The right-hand unit normal to $\vec{v}$ and $\vec{w}$



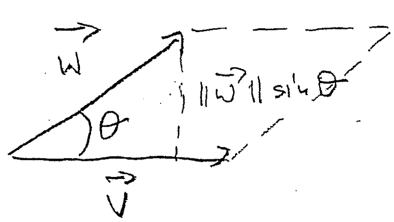
Curl the fingers of your right hand through the smaller of the two angles between $\vec{v}$ and $\vec{w}$ in the direction of $\vec{w}$. Your thumb is pointing toward $\vec{n}_R$.

$\vec{n}_R \perp \vec{v}$, $\vec{n}_R \perp \vec{w}$

$\|\vec{n}_R\| = 1$.

$\vec{v} \times \vec{w}$ points toward $\vec{n}_R$.

The area of a parallelogram

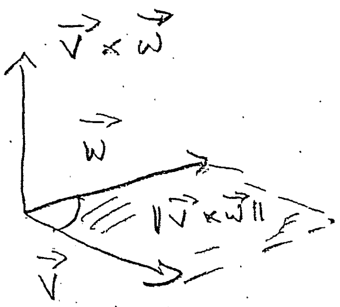


$$\boxed{A = \|\vec{w}\| \cdot \|\vec{v}\| \cdot \sin\theta}$$

① $\vec{v} \times \vec{w}$ geometrically

$$\vec{v} \times \vec{w} = (\|\vec{v}\| \, \|\vec{w}\| \sin\theta)\, \vec{n}_R ,$$

where $0 \le \theta \le \pi$ is the angle between $\vec{v}$ and $\vec{w}$, $\vec{n}_R$ the right-hand unit normal vector to $\vec{v}$ and $\vec{w}$.



$(\vec{v} \times \vec{w}) \perp \vec{v}$

$(\vec{v} \times \vec{w}) \perp \vec{w}$

$\vec{v}, \vec{w}, \vec{v} \times \vec{w}$ right-handed

$\|\vec{v} \times \vec{w}\| = A$.

② $\vec{v} \times \vec{w}$ algebraically

$$\vec{v} \times \vec{w} = (v_2 w_3 - v_3 w_2)\vec{i} + (v_3 w_1 - v_1 w_3)\vec{j} + (v_1 w_2 - v_2 w_1)\vec{k}.$$

Impossible to remember. Easy in terms of determinants.

Recall: 2×2 determinant:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - cb$$

3×3 determinant:

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

To compute $\vec{v} \times \vec{w}$:

$$\vec{v} \times \vec{w} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = \vec{i} \cdot (v_2 w_3 - v_3 w_2) - \vec{j}\,(v_1 w_3 - v_3 w_1) + \vec{k} \cdot (v_1 w_2 - v_2 w_1) =$$

$$= \vec{i} \begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix} - \vec{j} \begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix} + \vec{k} \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix}.$$

<u>Remark</u>: If $\vec{v} \parallel \vec{w}$, $\vec{v} \times \vec{w} = \vec{0}$

<u>Ex</u>: Let $\vec{v} = 2\vec{i} + \vec{j} - 2\vec{k}$, $\vec{w} = 3\vec{i} + \vec{k}$.

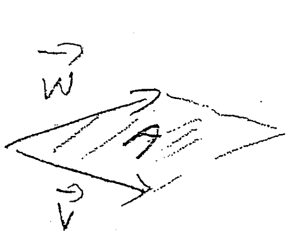
(a) Find $\vec{v} \times \vec{w}$

(b) Find the area of the parallelogram spanned by $\vec{v}$ and $\vec{w}$.

$$\vec{v} \times \vec{w} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 1 & -2 \\ 3 & 0 & 1 \end{vmatrix} = \vec{i} \begin{vmatrix} 1 & -2 \\ 0 & 1 \end{vmatrix} - \vec{j} \begin{vmatrix} 2 & -2 \\ 3 & 1 \end{vmatrix} + \vec{k} \begin{vmatrix} 2 & 1 \\ 3 & 0 \end{vmatrix} =$$

$$= \vec{i} - 8\vec{j} - 3\vec{k}$$

(b)

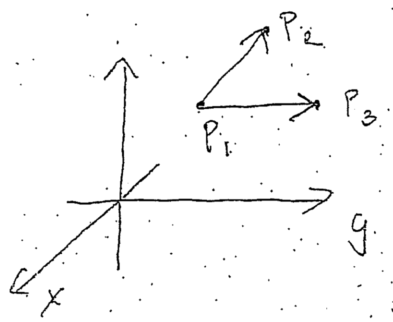


$$A = \| \vec{v} \times \vec{w} \| = \sqrt{1 + 64 + 9} \cong 8.6$$

<u>Ex</u>: Find an equation of the plane passing through points

$P_1 = (1, 1, 1)$, $P_2 = (0, 1, 2)$, $P_3 = (2, -5, 0)$.

Find a <u>unit</u> normal vector to the plane.



We have a point, we need a normal vector.
A normal will have to be $\perp$ to $\overrightarrow{P_1P_2}$ and $\overrightarrow{P_1P_3}$.

$$\vec{n} = (\overrightarrow{P_1P_2}) \times (\overrightarrow{P_1P_3})$$

$$\overrightarrow{P_1P_2} = (-1, 0, 1), \quad \overrightarrow{P_1P_3} = (1, -6, -1)$$

$$\vec{n} = \begin{vmatrix} \vec{i}, & \vec{j}, & \vec{k} \\ -1 & 0 & 1 \\ 1 & -6 & -1 \end{vmatrix} = \vec{i}\cdot 6 - \vec{j}\cdot 0 + \vec{k}\cdot 6$$

$$\vec{n} = 6\vec{i} + 6\vec{k}$$

i) Equation (using P_1):

$$6(x-1) + 6(z-1) = 0$$

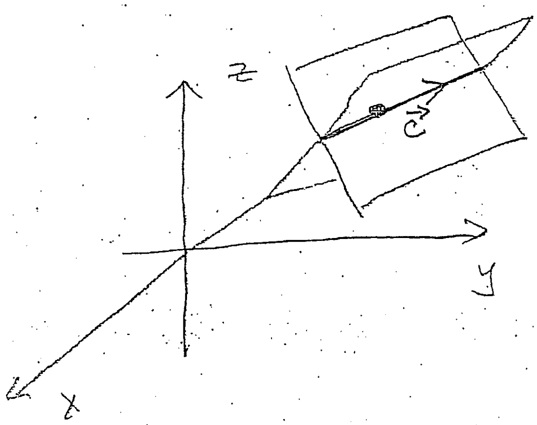
(b) $$\vec{u}_n = \frac{\vec{n}}{\|\vec{n}\|} = \frac{6}{6\sqrt{2}}\vec{i} + \frac{6}{6\sqrt{2}}\vec{k} = \frac{1}{\sqrt{2}}\vec{i} + \frac{1}{\sqrt{2}}\vec{k}.$$

$$\|\vec{n}\| = \sqrt{36+36} = 6\sqrt{2}$$

Ex: Find a vector parallel to the intersection of the planes

L_1: $2x - 3y + 5z = 2$

L_2: $4x + y - 3z = 7$



$\vec{c} \perp n_{L_1}$, $\vec{c} \perp n_{L_2}$

$n_{L_1} = (2, -3, 5)$

$n_{L_2} = (4, 1, -3)$

$$\vec{c} = n_{L_1} \times n_{L_2} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & -3 & 5 \\ 4 & 1 & -3 \end{vmatrix} =$$

$$= 4\vec{i} - (-26)\vec{j} + 14\vec{k} = \underline{4\vec{i} + 26\vec{j} + 14\vec{k}}.$$