

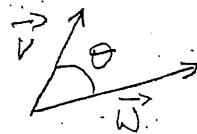
13.3 The Dot Product, cont'd

Recall for any:

$$\vec{v} = v_1 \vec{i} + v_2 \vec{j} + v_3 \vec{k}, \quad \vec{w} = w_1 \vec{i} + w_2 \vec{j} + w_3 \vec{k}$$

the dot product $\vec{v} \cdot \vec{w}$ can be defined algebraically or geometrically:

① $\vec{v} \cdot \vec{w} = v_1 w_1 + v_2 w_2 + v_3 w_3$



② $\vec{v} \cdot \vec{w} = \|\vec{v}\| \cdot \|\vec{w}\| \cos(\theta)$

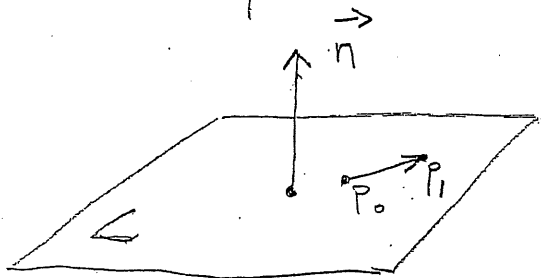
where $0 \leq \theta \leq \pi$ is the angle between \vec{v} and \vec{w} .

\vec{v} is perpendicular to \vec{w} , $\vec{v} \perp \vec{w}$, if and only if

$$\underline{\vec{v} \cdot \vec{w} = 0}$$

A Normal Vector to a Plane

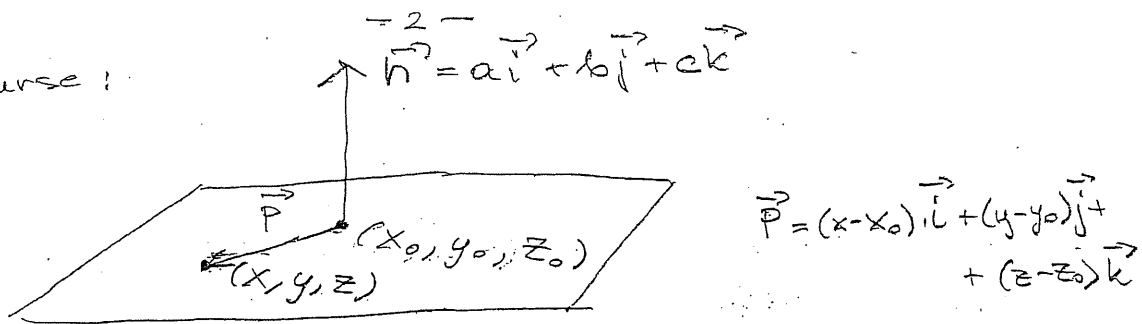
Def: \vec{n} is a normal vector to a plane L if \vec{n} is perpendicular to any vector $\vec{P_0 P_1}$ for any two points P_0, P_1 on the plane.



Remark: If $\vec{n} = a\vec{i} + b\vec{j} + c\vec{k}$ is normal to a plane L , $P_0 = (x_0, y_0, z_0)$ is a point on L , then the equation of L is:

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0.$$

Of course:



$$\vec{r} \cdot \vec{n} = a(x-x_0) + b(y-y_0) + c(z-z_0) = 0$$

We can rewrite:

$$ax + by + cz = d, \quad \vec{n} = a\vec{i} + b\vec{j} + c\vec{k}$$

Ex 1: Find the equation of the plane, L_0 , through $P_0 = (-2, 3, 2)$ and parallel to the plane

$$L_1: 3x + y + z = 4.$$

Sol. Two planes are parallel if their normal vectors are the same. Normal to L_1 is:

$$\vec{n} = 3\vec{i} + \vec{j} + \vec{k}, \quad P_0 = (-2, 3, 2).$$

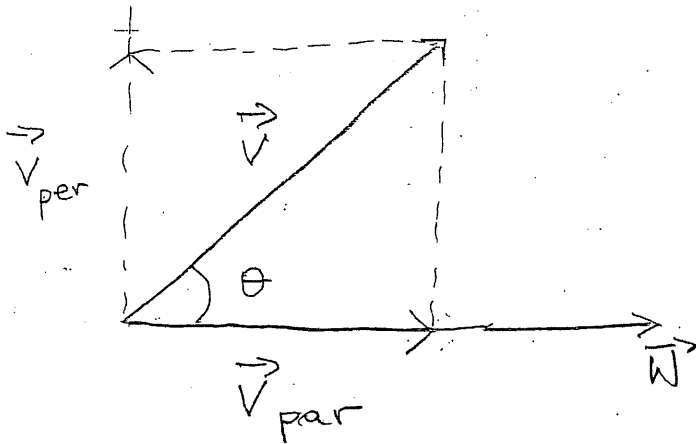
So:

$$L_0: 3(x+2) + (y-3) + (z-2) = 0$$

The dot product can be used to find parallel and perpendicular components of one vector with respect to another vector.

Very important in physics.

Let \vec{v} , \vec{w} be two vectors in \mathbb{R}^d .



Let $\vec{u} = \frac{\vec{w}}{\|\vec{w}\|}$ (The unit vector in the direction of \vec{w} .)

We have:

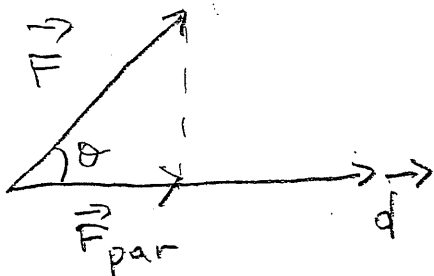
$$\|\vec{v}_{\text{par}}\| = \|\vec{v}\| \cdot \cos \theta = \|\vec{u}\| \|\vec{v}\| \cos \theta = \vec{v} \cdot \vec{u}$$

Thus:

$$\vec{v}_{\text{par}} = (\vec{v} \cdot \vec{u}) \vec{u}, \quad \vec{v}_{\text{per}} = \vec{v} - \vec{v}_{\text{par}}$$

\vec{v}_{par} - projection of \vec{v} onto \vec{w} .

Work: $W = F \cdot d$ if force and displacement are in the same direction.



$$W = \|\vec{F}_{\text{par}}\| \cdot \|\vec{d}\|$$

$$\|\vec{F}_{\text{par}}\| = \|\vec{F}\| \cdot \cos \theta = \frac{\vec{F} \cdot \vec{d}}{\|\vec{d}\|}$$

So

$$\underline{W = \vec{F} \cdot \vec{d}}$$

Ex: Write $\vec{a} = 3\vec{i} + 2\vec{j} - 6\vec{k}$ as the sum of two vectors, one parallel, and one perpendicular to $\vec{d} = 2\vec{i} - 4\vec{j} + \vec{k}$.

Sol:

$$\vec{a} = \vec{a}_{\text{par}} + \vec{a}_{\text{per}}$$

$$\vec{a}_{\text{par}} = \left(\vec{a} \cdot \frac{\vec{d}}{\|\vec{d}\|} \right) \cdot \frac{\vec{d}}{\|\vec{d}\|}, \quad \vec{u} = \frac{\vec{d}}{\|\vec{d}\|}$$

$$\|\vec{d}\| = \sqrt{4 + 16 + 1} = \sqrt{21}$$

$$\vec{u} = \frac{2}{\sqrt{21}}\vec{i} - \frac{4}{\sqrt{21}}\vec{j} + \frac{1}{\sqrt{21}}\vec{k}$$

$$\vec{a}_{\text{par}} = \left(\frac{6}{\sqrt{21}} - \frac{8}{\sqrt{21}} - \frac{6}{\sqrt{21}} \right) \cdot \left(\frac{2}{\sqrt{21}}\vec{i} - \frac{4}{\sqrt{21}}\vec{j} + \frac{1}{\sqrt{21}}\vec{k} \right) =$$

$\underbrace{\hspace{10em}}_{\vec{a} \cdot \vec{u}}$

$$= \frac{-16}{21}\vec{i} + \frac{32}{21}\vec{j} - \frac{8}{21}\vec{k} = \vec{a}_{\text{par}} = \frac{8}{21}\vec{d}$$

$$\vec{a}_{\text{per}} = \vec{a} - \vec{a}_{\text{par}} = (3\vec{i} + 2\vec{j} - 6\vec{k}) - \left(\frac{-16}{21}\vec{i} + \frac{32}{21}\vec{j} - \frac{8}{21}\vec{k} \right) =$$

$$= \frac{79}{21}\vec{i} + \frac{10}{21}\vec{j} - \frac{118}{21}\vec{k} = \vec{a}_{\text{per}}$$

$$\vec{a} = \vec{a}_{\text{par}} + \vec{a}_{\text{per}}$$