# 13.3 The Dot Product - Part I

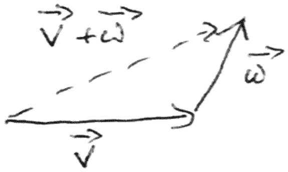
In the previous video, we defined the following operations on vectors:

$\vec{v}+\vec{w}$, $\vec{v}-\vec{w}$, $\alpha\vec{v}$, $\|\vec{v}\|$.

Each operation we defined in two ways: geometric and in components.

For example:

$\vec{v}+\vec{w}$



or equivalently $\vec{v}=v_1\vec{i}+v_2\vec{j}+v_3\vec{k}$, $\vec{w}=w_1\vec{i}+w_2\vec{j}+w_3\vec{k}$:

$$\vec{v}+\vec{w}=(v_1+w_1)\vec{i}+(v_2+w_2)\vec{j}+(v_3+w_3)\vec{k}$$

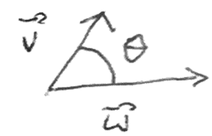
How about multiplication of vectors? There are two types of multiplication of vectors: the dot product and the cross product.

**Def:** Let $\vec{v}=v_1\vec{i}+v_2\vec{j}+v_3\vec{k}$ and $\vec{w}=w_1\vec{i}+w_2\vec{j}+w_3\vec{k}$ be given. We define the dot product $\vec{v}\cdot\vec{w}$ as follows:

$$\vec{v}\cdot\vec{w}=v_1w_1+v_2w_2+v_3w_3$$

or equivalently:

$$\vec{v}\cdot\vec{w}=\|\vec{v}\|\|\vec{w}\|\cos\theta$$



where $0\le\theta\le\pi$ is the angle between $\vec{v}$ and $\vec{w}$.

The algebraic and the geometric definitions of the dot product are indeed equivalent. It can be proved using the Law of Cosines. We will need both definitions to use the dot product effectively.

**Properties of the Dot Product:**

(a) $\vec{v}\cdot\vec{w}=\vec{w}\cdot\vec{v}$ (b) $\vec{v}\cdot(\lambda\vec{w})=(\lambda\vec{v})\cdot\vec{w}=\lambda(\vec{v}\cdot\vec{w})$

(c) $\vec{v}\cdot(\vec{u}+\vec{w})=\vec{v}\cdot\vec{u}+\vec{v}\cdot\vec{w}$

(d) $\vec{v}\cdot\vec{v}=\|\vec{v}\|^2$

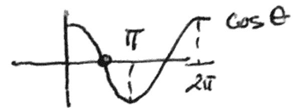
(e) $\vec{v}$ is perpendicular to $\vec{w}$ (denoted $\vec{v}\perp\vec{w}$) if and only if $\vec{v}\cdot\vec{w}=0$.

Properties (a)-(d) follow easily from the algebraic definition of the dot product. Why is (e) true?

Take $\vec{v} \neq \vec{0}$, $\vec{w} \neq \vec{0}$. What does it mean that $\vec{v} \cdot \vec{w} = 0$?

$$\vec{v} \cdot \vec{w} = \underset{\neq 0}{\|\vec{v}\|}\, \underset{\neq 0}{\|\vec{w}\|} \cos\theta = 0 \iff \cos\theta = 0 \iff \theta = \frac{\pi}{2}$$

$$0 \leq \theta \leq \pi$$




So the angle between $\vec{v}$ and $\vec{w}$ is $\frac{\pi}{2}$ or $90^\circ$.

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$\vec{v} \perp \vec{w}$, $\vec{v}$ perpendicular to $\vec{w}$, $\vec{v}$ and $\vec{w}$ orthogonal, $\vec{v}$ normal to $\vec{w}$

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Having both definitions is what makes the dot (scalar) product useful.

Ex: Find the angle (in degrees) between the vectors
$$\vec{v} = 2\vec{i} - \vec{j} + \vec{k} \quad \text{and} \quad \vec{w} = \vec{i} + \vec{j} - 3\vec{k}.$$

Solution: We are looking for the angle $\theta$ that appears in the dot product. We know
$$\vec{v} \cdot \vec{w} = \|\vec{v}\|\|\vec{w}\| \cos\theta.$$
We have $\vec{v}$ and $\vec{w}$ in terms of their components so we can calculate $\vec{v} \cdot \vec{w}$, $\|\vec{v}\|$ and $\|\vec{w}\|$:
$$\vec{v} \cdot \vec{w} = 2 \cdot 1 + (-1) \cdot 1 + 1 \cdot (-3) = -2$$
$$\|\vec{v}\| = \sqrt{2^2 + (-1)^2 + (1)^2} = \sqrt{6}, \quad \|\vec{w}\| = \sqrt{(1)^2 + (1)^2 + (-3)^2} = \sqrt{11}.$$

Hence:
$$-2 = \sqrt{6} \cdot \sqrt{11} \cos\theta$$
$$\cos\theta = -\frac{2}{\sqrt{6}\sqrt{11}}, \quad \theta = \cos^{-1}\left(-\frac{2}{\sqrt{66}}\right) = 104.25^\circ$$

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In the next video, we will look at other applications of the dot product.

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Please remember: The dot product of two vectors is a number!

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Given two points, $P_0 = (x_0, y_0, z_0)$, $P_1 = (x_1, y_1, z_1)$, we define the displacement vector $\overrightarrow{P_0P_1} = (x_1 - x_0)\vec{i} + (y_1 - y_0)\vec{j} + (z_1 - z_0)\vec{k}$.