12.5 Functions of Three Variables

We already talked about functions of three variables:

$$w = f(x, y, z).$$

The graph of a function of three variables is the collection of all quadruples:

$$(x, y, z, f(x, y, z))$$

for all triples (x, y, z) in the domain of f. The domain is the subset of the xyz-space — the 3D space. The graph is a subset of the $xyzw$-space — the 4D space. Hence, we cannot visualize functions of three variables through their graphs.

We can visualize functions of three variables through **level surfaces**.

Level surfaces are analogous to level curves for a function of two variables $z = f(x, y)$. How were level curves (also called contours) obtained? We took a constant c and considered the curve on the xy-plane of all points (x, y) for which:

$$f(x, y) = c.$$

The level curve corresponding to the function value c is the collection of all points (x, y) for which the value of z — the elevation on the graph — is c. The graph of $z = f(x, y)$ is in 3D space. Contours are on the xy-plane so we drop one dimension down.

We do something similar for a function of three variables $w = f(x, y, z)$. We take a constant c — the function value c — and consider the surface in the xyz-space of all points (x, y, z) for which

$$f(x, y, z) = c.$$

We call this surface the level surface for f corresponding to the function value c.

Level Surfaces for $f(x, y, z)$

A level surface (or a level set) of a function of three variables $f(x, y, z)$ is a surface in the xyz-space of the form:

$$f(x, y, z) = c$$

where c is a constant. The function f can be represented by the family of level surfaces by allowing c to vary.

Example 1. The temperature, T, in °C, at a point (x, y, z) is given by:

$$T = f(x, y, z) = x^2 + y^2 + z^2.$$

What do the level surfaces of the function f look like and what do they mean in terms of temperature?

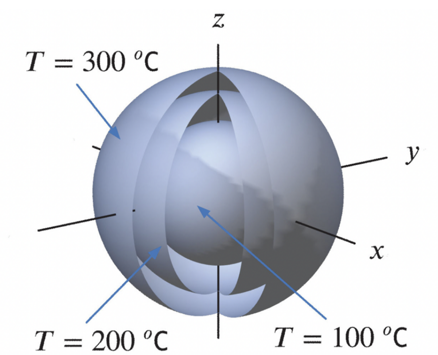
Solution. Choose a few values c for T and see what the corresponding level surfaces are. Take $c = 100, 200, 300$; that is, the temperature values $c = 100°\text{C}$, $200°\text{C}$, $300°\text{C}$. What are surfaces $T = f(x, y, z) = c$ for our values of c? They are surfaces in the xyz-space satisfying the equations:

$$x^2 + y^2 + z^2 = 100,$$

$$x^2 + y^2 + z^2 = 200,$$

$$x^2 + y^2 + z^2 = 300.$$

They are, of course, spheres centered at the origin with radii: 10, $\sqrt{200} \approx 14.14$, and $\sqrt{300} \approx 17.32$. Here is a picture:



What we see is that temperature T at (x, y, z) depends only on the distance of the point (x, y, z) from the origin and is equal to the square of this distance. The temperature is constant on every sphere centered at the origin and it is increasing faster and faster as we move away from the origin (as it takes a smaller and smaller change in radius to get the increase of 100 in temperature).

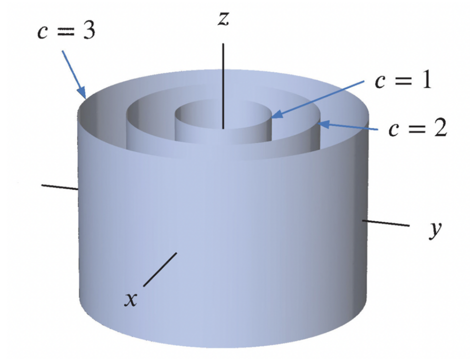
The level surface "diagram" certainly gives us an insight into the temperature distribution.

Example 2. What do the level surfaces of $f(x, y, z) = x^2 + y^2$ look like?

Solution. Take a function value c. The corresponding level surface is:

$$x^2 + y^2 = c.$$

For c negative the surface is empty. For $c = 0$, the surface consists of points $(0, 0, z)$; that is, the z-axis. For $c > 0$, the surface is the cylinder about the z-axis with radius $\sqrt{c}$:



The level surfaces form concentric cylinders about the z-axis.

Example 3. What do the level surfaces of $g(x, y, z) = z - y$ look like?

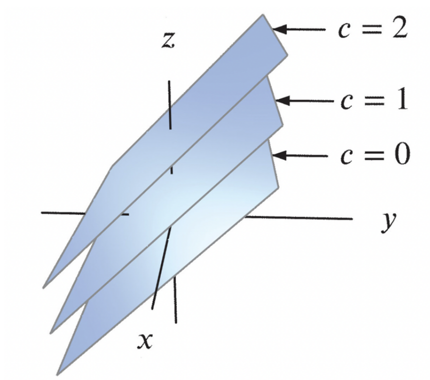
Solution. Take a function value c. The corresponding level surface is:

$$z - y = c.$$

This is a plane in the xyz-space. We can see the surface as the graph of a linear function $z = f(x, y)$ of two variables:

$$z = y + c.$$

Here is a picture for the function values $c = 0, 1, 2$:



Later on it will be important to recognize that a given surface in the xyz-space is a level surface of a function of three variables.

Example 4. Consider a surface S:

$$S : \quad x^2 + y^3 - z + 5 = 0.$$

Find a function $f(x, y, z)$ for which the surface S is a level surface.

Solution. There are many possible answers to this problem. The most obvious choice is the function:

$$f(x, y, z) = x^2 + y^3 - z + 5.$$

Then S is the level surface for f corresponding to the function value $c = 0$.

$$f(x, y, z) = x^2 + y^3 - z + 5 = 0.$$

We could choose:

$$h(x, y, z) = x^2 + y^3 - z.$$

Then S is the level surface for h corresponding to the function value -5:

$$h(x, y, z) = x^2 + y^3 - z = -5$$

Example 5. Consider a surface S:

$$S : \quad x^2 + y^3 - z + 5 = 0.$$

Is there a function of two variables $z = g(x, y)$ whose graph is S?

Solution. In general the answer to such problems can sometimes be yes and sometimes be no. In this example we can solve for z and rewrite the equation for S in the form:

$$z = x^2 + y^3 + 5.$$

So S is the graph of the function $z = g(x, y) = x^2 + y^3 + 5$.
