12.4 Linear Functions of Two Variables

Linear functions of one variable, $y = mx + b$, are functions whose graphs on the xy-plane are straight lines. Linear functions are important in single-variable calculus as they provide a simple local approximation for more complicated functions at any point where the graph of the function has a tangent line. The situation is analogous for functions of two variables. Linear functions of two variables are functions whose graphs are planes in the xyz-space. Whenever a more complicated function of two variables has a tangent plane at a point, the corresponding linear function provides a local linear approximation to that function around the point.

Linear Function of Two Variables

A function $z = f(x, y)$ is called linear if it can be written in the form:

$$f(x, y) = mx + ny + c$$

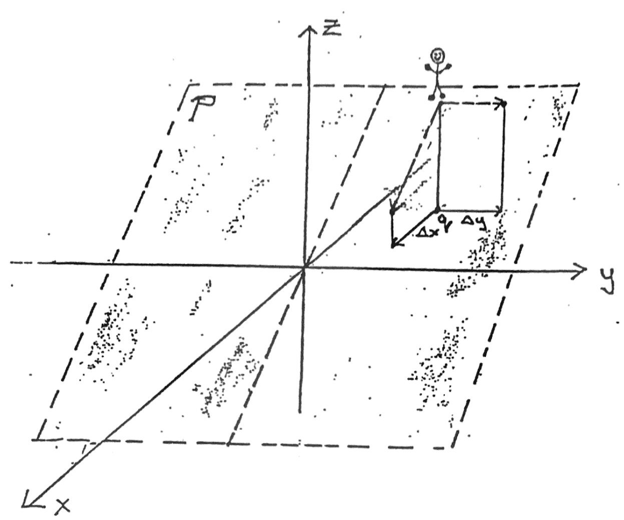
where m, n, and c are constants. The constant m is called the slope in the x-direction, n is called the slope in the y-direction, $c = f(0, 0)$ is the z-intercept. The graph of every linear function is a plane in the xyz-space and any function $f(x, y)$ whose graph is a plane is linear.

The geometric meaning of m and n is exactly what they names suggest.

Example 1. Consider a function

$$z = f(x, y), \quad f(x, y) = -x.$$

The graph of the function is the plane $z = -x$. Let's call the plane P. Taking the cross-section of P with the plane $y = 0$, we obtain the line $z = -x$ in the xz-plane. As the function is constant in y, we obtain P by sliding the line along the y-axis:



The function $f(x, y)$ is linear with:

$$m = -1, \qquad n = 0, \qquad c = 0.$$

The graph shows the meaning of the slopes. The stick-figure man is standing at a point on the plane P. If the man walks parallelly to the xz-plane, without any displacement in the y-direction, the man will be walking with the slope -1. If the man walks parallelly to the yz-plane, without any displacement in the x-direction, the man will be walking with the slope 0. The z-intercept, c, is the value of z at the point where the plane intersects the z-axis; that is, $c = 0$.

The main property of a linear function of one variable, $y = mx + b$, is the constancy of the rate of change of the dependent variable with respect to the independent variable. Linear functions of two variables change at a constant rate provided we move in a fixed direction. In particular, the following is true.

Slopes and Changes

Consider a linear function:

$$f(x, y) = mx + ny + c.$$

- When y is fixed and x changes, to equal changes Δx in x there correspond equal changes Δz in z and:

  $$\Delta z = m\Delta x.$$

  Hence, for y fixed:

  $$m = \frac{\Delta z}{\Delta x}.$$

- When x is fixed and y changes, to equal changes Δy in y there correspond equal changes Δz in z and:

  $$\Delta z = n\Delta y.$$

  Hence, for x fixed:

  $$n = \frac{\Delta z}{\Delta y}.$$

All the properties above follow from algebraic properties of linear functions. Take two points (x_1, y_1) and (x_2, y_2) on the xy-plane and the corresponding values of a linear function $z = mx + ny + c$:

$$z_1 = mx_1 + ny_1 + c, \qquad z_2 = mx_2 + ny_2 + c.$$

Denote the changes as $\Delta x = x_2 - x_1$, $\Delta y = y_2 - y_1$, and $\Delta z = z_2 - z_1$. Then:

$$\Delta z = z_2 - z_1 = (mx_2 + ny_2 + c) - (mx_1 + ny_1 + c) = m(x_2 - x_1) + n(y_2 - y_1) = m\Delta x + n\Delta y.$$

When y is fixed and doesn't change, $\Delta y = y_2 - y_1 = 0$. Hence:

$$\Delta z = m\Delta x.$$

When x is fixed and doesn't change, $\Delta x = x_2 - x_1 = 0$. Hence:

$$\Delta z = n\Delta y.$$

The form $f(x, y) = mx + ny + c$ of a linear function is useful if we have slopes and the vertical intercept. If we have the slopes m and n and a point on the graph of the function, the following "point-slope" form of a linear function or a plane is easier to use.

Point-Slope Form of a Plane and a Linear Function

If a plane has slope m in the x-direction, slope n in the y-direction, and passes through the point (x_0, y_0, z_0), then its equation is:

$$z = z_0 + m(x - x_0) + n(y - y_0).$$

The plane is the graph of the linear function:

$$f(x, y) = z_0 + m(x - x_0) + n(y - y_0).$$

How to recognize that a function $z = f(x, y)$ given numerically is linear?

A linear function can be recognized from its table by the following features:

- Each row and each column is linear.
- All the rows have the same slope.
- All the columns have the same slope (although the slope of the rows and the slope of the columns are generally different).

Example 2. Could a table of values correspond to a linear function? If yes, find a formula for the function.

(a) $f(x, y)$:

		y		
		0	1	2
x	0	5	7	9
	1	6	9	12
	2	7	11	15

(b) $g(x, y)$:

x\y	10	20	30	40
100	3	6	9	12
200	2	5	8	11
300	1	4	7	10
400	0	3	6	9

Solution. (a) Let's look at rows first. In each row x is fixed and y changes. Changes in y corresponding to the consecutive values are equal: $\Delta y = 1 - 0 = 2 - 1 = 1$. Let's check the corresponding changes in z. In row 1, $\Delta z = 7 - 5 = 9 - 7 = 2$. So row 1 is linear with slope $\frac{\Delta z}{\Delta y} = \frac{2}{1} = 2$. In row 2, the changes in z are $\Delta z = 9 - 6 = 12 - 9 = 3$. So row 2 is linear with slope $\frac{\Delta z}{\Delta y} = \frac{3}{1} = 3$. Since the slopes in the y-direction corresponding to two different fixed values of x are different, the table does not correspond to a linear function $f(x, y)$. In a linear function, for each fixed x, the slope in the y-direction must be the same.

(b) Let's look at rows. In each row x is fixed and y changes through equally spaced values: $\Delta y = 20 - 10 = 30 - 20 = 40 - 30 = 10$. In row 1, $\Delta z = 6 - 3 = 9 - 6 = 12 - 9 = 3$. We look at row 2, 3 and 4, and see that in each row changes in z are the same, $\Delta z = 3$. We conclude that the function $g(x, y)$ may be linear.

We have to check all columns. In each column y is fixed and x changes through equally spaced values: $\Delta x = 200 - 100 = 300 - 200 = 400 - 300 = 100$. In column 1, z changes by -1 at each step: $\Delta z = 2 - 3 = 1 - 2 = 0 - 1 = -1$. We check column 2, 3, and 4 and see that in each column changes in z corresponding to $\Delta x = 100$ are equal and equal to -1. Hence, the table does correspond to a linear function; that is, $g(x, y)$ is linear.

To find the formula for $g(x, y)$ we easily find both slopes:

$$m = \frac{\Delta z}{\Delta x} = \frac{-1}{100} = -0.01,$$

$$n = \frac{\Delta z}{\Delta y} = \frac{3}{10} = 0.3.$$

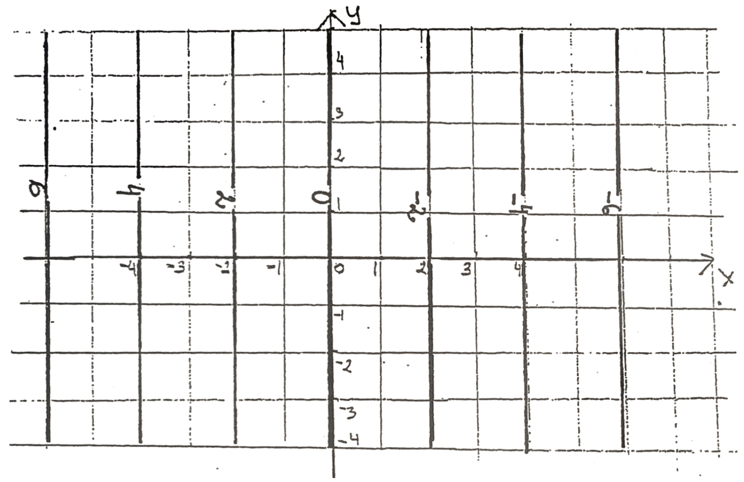
We don't have the value for c given directly as we don't have $f(0, 0)$ given. We use point-slope form with any one of the 16 points on the graph given in the table. Take, for example, $(x_0, y_0, z_0) = (100, 10, 3)$. We obtain:

$$g(x, y) = -0.01(x - 100) + 0.3(y - 10) + 3.$$

The latter expression simplifies to:

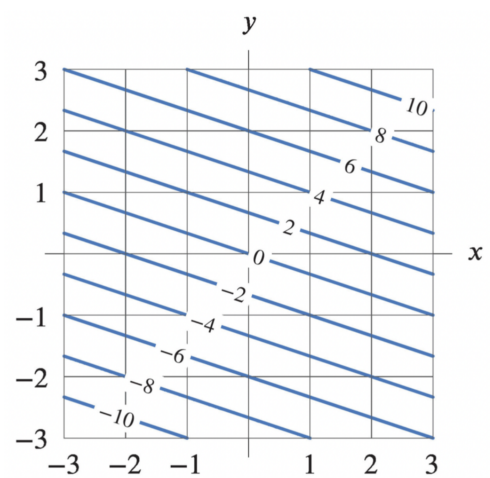
$$g(x, y) = -0.01x + 0.3y + 1.$$

The graph of each linear function is a plane. How does a contour map of a linear function look? In Example 1 we looked at the function $z = -x$ and its graph P. It is clear from the graph that the intersection of P with any horizontal plane is a line in P parallel to the y axis. It is clear that the contour map for z values $-6, -4, -2, 0, 2, 4, 6$ is:



It turns out that a contour diagram (corresponding to equally spaced elevations) of every linear function is a collection of parallel, equally-spaced lines whose elevation increase one way and decrease the opposite way when we move perpendicularly to the lines.

Example 3. Find a formula for a linear function $h(x, y)$ whose contour map is below.



Solution. From the contour map, we can read the exact coordinates for many points on the graph $z = h(x, y)$. For example, the contour corresponding to the elevation 0 passes through the point $(x, y) = (0, 0)$. Hence, $h(0, 0) = 0$ which gives $c = 0$. To find m, notice that for y fixed at $y = 0$, we have the contour corresponding to the elevation 0 passing through $x = 0$ and the contour corresponding to the elevation 2 passing through $x = 2$. Hence the points $(0, 0, 0)$ and $(2, 0, 2)$ are on the graph. We obtain:

$$m = \frac{\Delta z}{\Delta x} = \frac{2 - 0}{2 - 0} = 1.$$

Similarly, looking at the y-axis, we see that the contour corresponding to the elevation 6 passes through $y = 2$ and the contour corresponding to the elevation 0 through $y = 0$. Hence:

$$n = \frac{\Delta z}{\Delta y} = \frac{6 - 0}{2 - 0} = 3.$$

The function is:

$$h(x, y) = x + 3y.$$