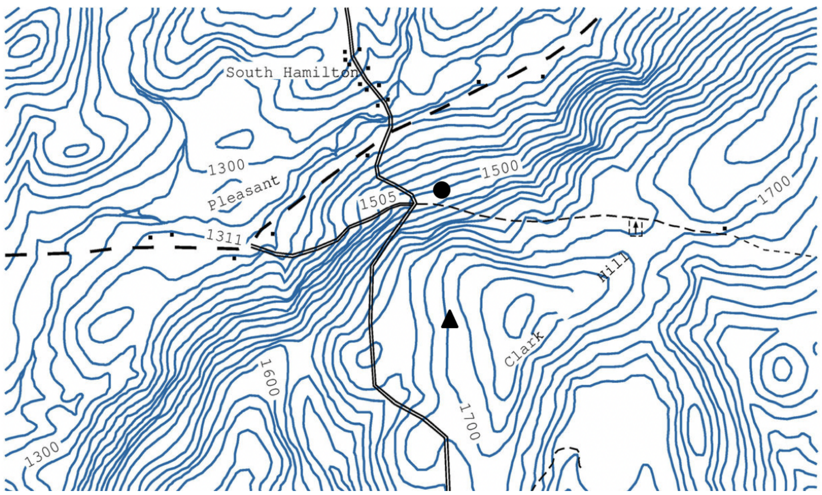
12.3 Contour Diagrams

One way to visualize functions of two variables $z = f(x, y)$ is through their graphs. Another type of visualization is through their contour diagrams or contour maps. A contour diagram is, in essence, a "topographical map" of the graph of $z = f(x, y)$. A topographical map is a two-dimensional visualization of three-dimensional terrain through the so-called **level curves** or **contours** corresponding to points of equal elevation.

Example 1. Here is a map of the region near South Hamilton, NY:



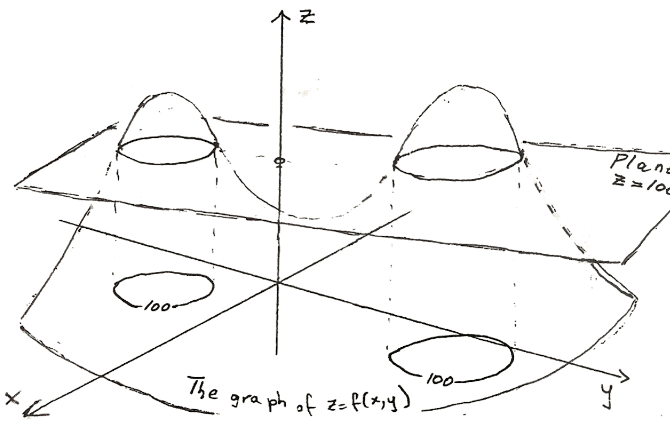
(a) If you are standing where the black dot is, how high are you? What do you see when you look south? If you start walking toward the top of Clark Hill, are you walking steeply or not?

(b) If you are standing at the black triangle and start walking toward the top of Clark Hill, are you walking steeply or not? What do you see from the top of Clark Hill when you look northeast?

Solution. (a) You are standing on the contour corresponding to the elevation 1500 feet. When you look south, you see a hill — Clark Hill. On most topographical maps contours correspond to equally spaced elevations. There are 10 contours between the contour corresponding to the elevation of 1500 feet and the contour corresponding to the elevation of 1700 feet. So consecutive elevations are 20 feet apart. When you walk toward Clark Hill, your are walking steeply as contours are very close together. That means that a small horizontal displacement produces a large increase in elevation.

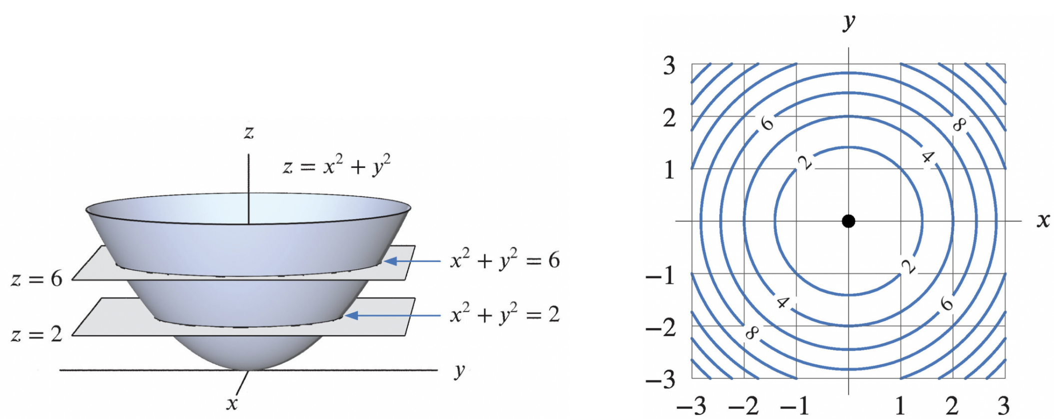
(b) Contours are far apart, so your ascent is gradual. You are not walking steeply. From the top of Clark Hill, you see another hill and a valley between two hills — a saddle.

You create a contour diagram corresponding to a function $z = f(x, y)$ by creating a topographical map of its graph. You choose equally spaced elevations $z = c$ for a bunch of values c, you find points on the graph for each elevation $z = c$, and then you project the curves on the graph onto the xy-plane. In the example below, we find the contour corresponding to the elevation $z = 100$. The contour happens to consist of two separate pieces:



Below are contour maps for some common graphs.

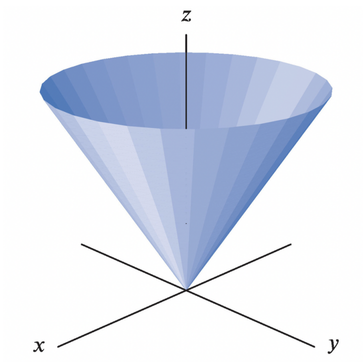
Example 2. Here is a paraboloid $z = x^2 + y^2$ and its contour map:

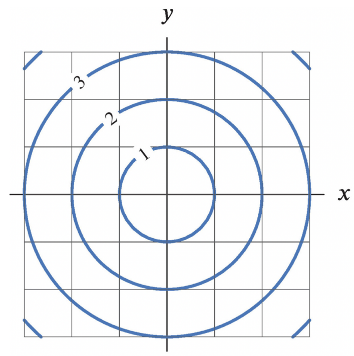


What is the shape of the level curves? How is the steepness of the paraboloid reflected in the contour map?

Solution. Level curves are circles as the curve $x^2 + y^2 = c$ is a cricle. The paraboloid is getting steeper and steeper so the contours are getting closer and closer together for higher and higher elevations.

Example 3. Here is a cone $z = \sqrt{x^2 + y^2}$ and its contour map:

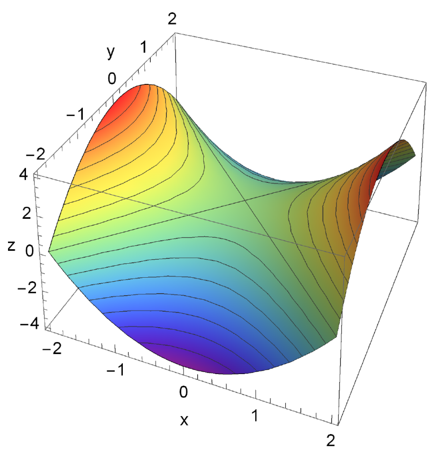


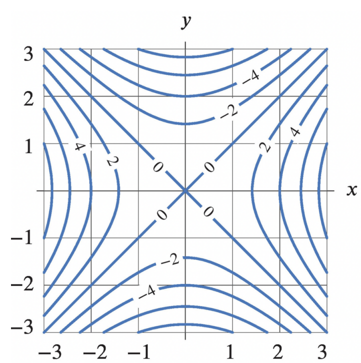


What is the shape of the level curves? How is the steepness of the cone reflected in the contour map?

Solution. Level curves are circles as the curve $\sqrt{x^2 + y^2} = c$ is a circle. The cone is of constant steepness so the contours corresponding to equally spaced elevations are equally spaced.

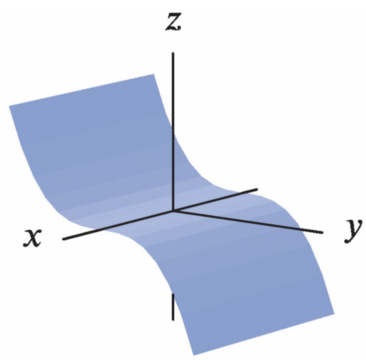
Example 4. A contour diagram of the saddle $z = x^2 - y^2$ has a very characteristic appearance:

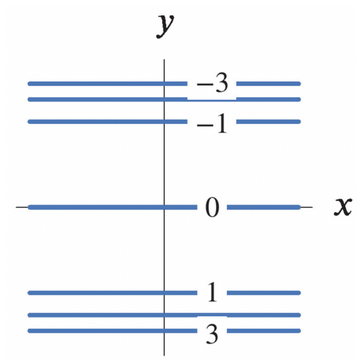




Imagine standing in the middle of the diagram. If you walk north or south, you will be crossing contours corresponding to lower and lower elevations; you are going down. If you walk west or east, you are going up.

Example 5. The graph and a contour diagram of the function $z = f(x, y)$ is given below.





Can you give a possible formula for $f(x, y)$?

Solution. Both the graph and the contour map indicate that $f(x, y)$ doesn't change when x changes. The values $f(x, y)$ depend only on y. The cross-section of the graph with the plane $x = 0$ indicates that a **possible** formula is $f(x, y) = -y^3$.

You can sketch a contour map for a function $f(x, y)$ given a formula for the function purely algebraically without graphing the function.

Example 6. Draw and label contours of the function $z = f(x, y) = y - x^2$ corresponding to function values $c = -1, 0, 1, 2$. What kind of curves are the contours?

Solution. For each constant c, the curve on the xy-plane representing the contour along which $f(x, y) = c$ satisfies the equation:

$$y - x^2 = c.$$

Equivalently:

$$y = x^2 + c.$$

This is the standard parabola in the xy-plane shifted vertically by c. The parabolas for the given values c are:

$$y = x^2 - 1, \qquad y = x^2, \qquad y = x^2 + 1, \qquad y = x^2 + 2.$$

The contour map with y-intercepts and elevations marked is: