12.2 Graphs of Functions of Two Variables, Surfaces in xyz-Space

As we know, graphs of functions of two variables are surfaces in the xyz-space. In this section, we will look at some important graphs as well as at some important surfaces that are not graphs of functions. We will draw a few graphs by hand. Of course most of the time we graph functions of two variables using graphing software like Mathematica or devices like graphing calculators. It is a good idea, though, to graph a few functions by hand to get a feel for such graphs.

Graphs of Functions of Two Variables — Examples

Example 1. Draw by hand the graph of the function $z = f(x, y)$, $f(x, y) = x^2 + y^2$.

Solution. We want to graph the surface given by the equation $z = x^2 + y^2$. The standard technique is to look at **cross-sections** — intersections — of the surface with planes parallel to the coordinate planes; that is, planes with equations $x = const$, $y = const$, and $z = const$.

What is the cross-section of the graph $z = x^2 + y^2$ with the plane $x = 0$; that is, with the yz-plane? The cross-section is a curve in the yz-plane which consists of the points (x, y, z) for which both conditions hold:

$$z = x^2 + y^2 \text{ and } x = 0.$$

Hence, the cross-section is the curve

$$z = y^2 = f(0, y)$$

in the yz-plane. The cross-section is the standard parabola in the yz-plane.

What is the cross-section of the graph with the plane $y = 0$; that is, with the xz-plane? It is of course the parabola

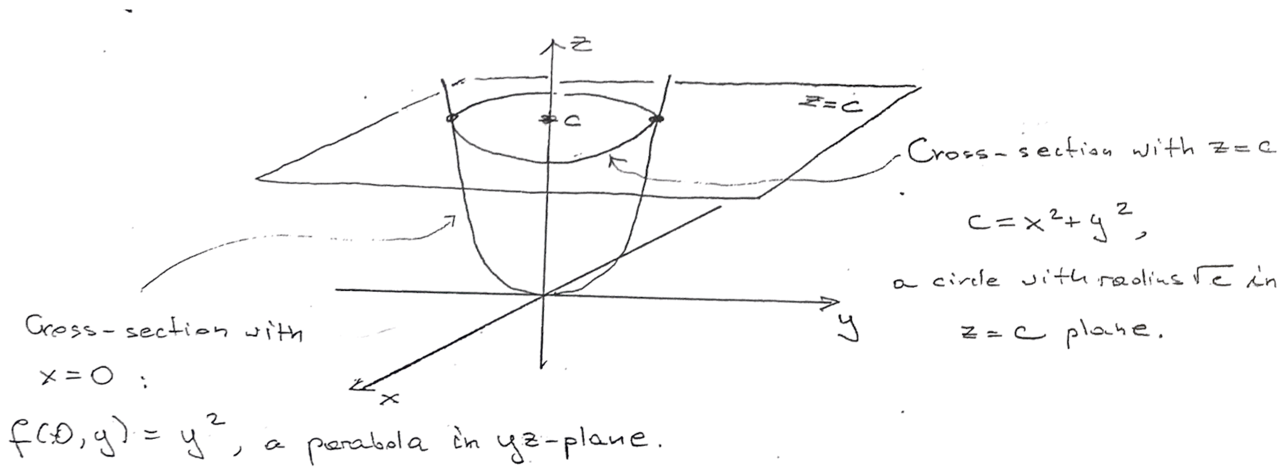
$$z = x^2 = f(x, 0)$$

in the xz-plane.

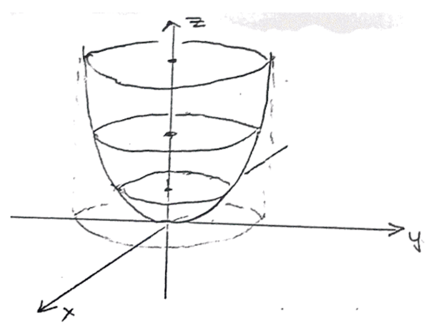
Those two parabolas do not tell us what the graph is. The key to this example is to take cross-sections with horizontal planes $z = c$ for any constant $c \geq 0$. Every such cross-section is a curve in the plane $z = c$ with the equation:

$$c = x^2 + y^2.$$

The curve is, of course, a circle in the $z = c$ plane centered at $(0, 0, c)$ with radius $\sqrt{c}$. The graph of the function $f(x, y) = x^2 + y^2$ is therefore circularly symmetric about the z-axis:

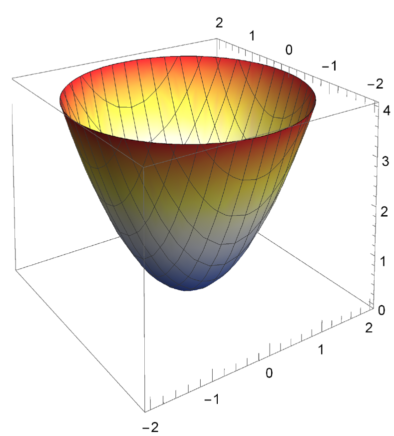
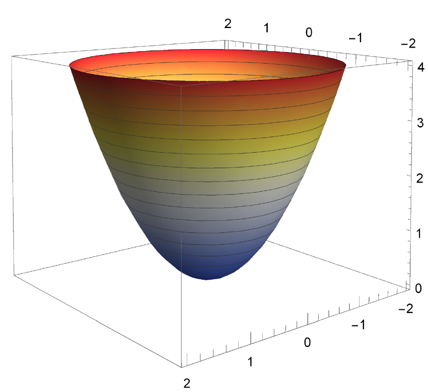


We obtain the graph of $z = f(x, y) = x^2 + y^2$ by revolving the parabola $z = y^2$ in the yz-plane about the z-axis:



The graph is a **paraboloid** in the xyz-space.

Here are the pictures of the paraboloid drawn by Mathematica. The first shows cross-sections of the graph with the planes $x = const$ and $y = const$, the second with the planes $z = const$:

Example 2. Draw by hand the graph of the function $z = f(x, y)$, $f(x, y) = x^2 - y^2$.

Solution. We need to draw the surface $z = x^2 - y^2$. Let's try a few cross-sections and see what may give us an insight into the graph of the function. The cross-section with $y = 0$, $z = f(x, 0)$ is the parabola:

$$z = x^2$$

in the xz-plane; that is, in the $y = 0$ plane. Let's call this parabola P_1. Take the cross-section with $x = 0$:

$$z = -y^2.$$

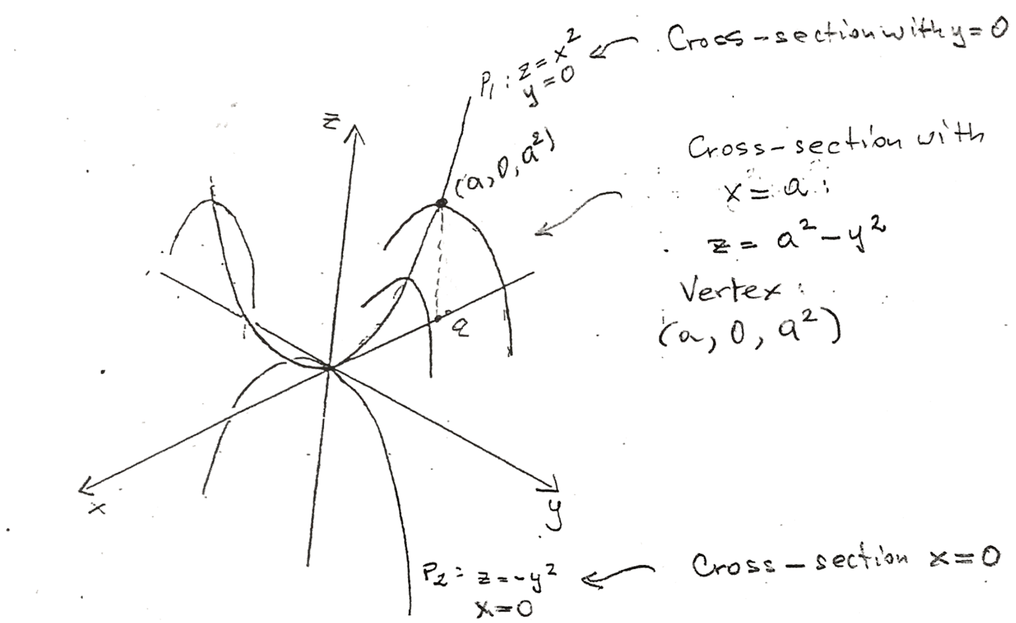
This is the upside down parabola in the yz-plane. Let's call this parabola P_2. All of that doesn't tell us much. What if we try intersections with horizontal planes $z = c$?

$$c = x^2 - y^2.$$

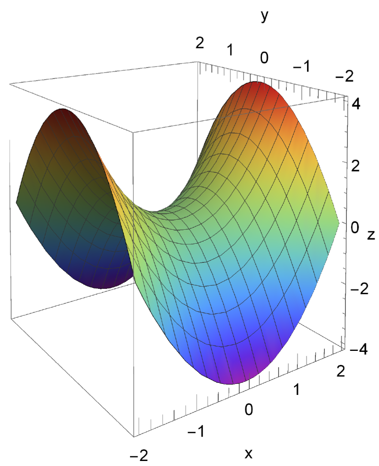
We get some kind of hyperbolas. Let's try intersections with planes of the form $x = a$ for any constant a; that is, intersections with planes parallel to the yz-plane:

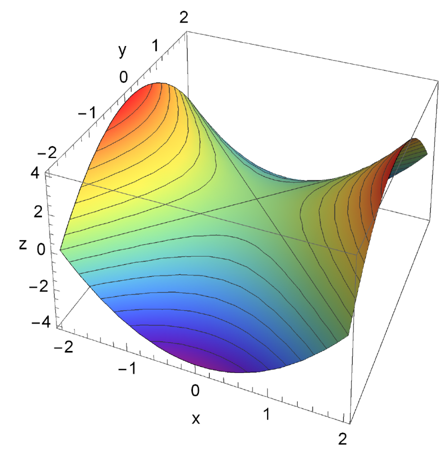
$$z = a^2 - y^2.$$

This is an upside down parabola P_2 moved to the plane $x = a$ with its vertex at $y = 0$ lifted a^2 units up. Hence, the vertex is at the point $(a, 0, a^2)$. We see that the vertex of the parabola on the $x = a$ plane is on the parabola P_1 and this is true for every constant a. Hence, the surface is obtained by sliding the parabola P_2 — sliding its vertex — along the parabola P_1.



We obtain a surface called a **saddle**. Here are pictures generated by Mathematica, one showing cross-sections with $x = const$ and $y = const$, the other cross-sections with horizontal planes $z = const$.





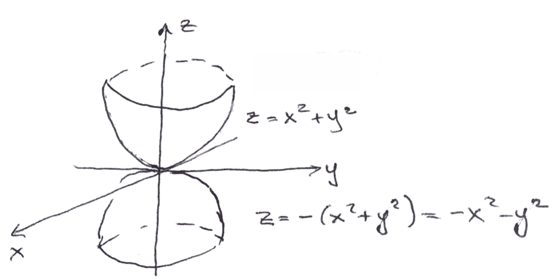
If we have one graph we can get many others by simple transformations.

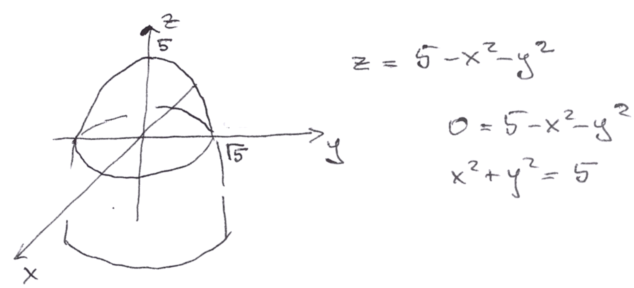
Example 3. Sketch by hand graphs of the following functions $z = f(x, y)$ and $z = g(x, y)$:

$$f(x, y) = -x^2 - y^2,$$
$$g(x, y) = 5 - x^2 - y^2.$$

What is the intersection of $z = g(x, y)$ with the xy-plane?

Solution. We use the paraboloid $z = x^2 + y^2$. Notice that $z = -x^2 - y^2 = -(x^2 + y^2)$ is the upside down paraboloid, the paraboloid $z = x^2 + y^2$ flipped about the xy-plane. Indeed, the only thing that changes is the sign of z. The surface $z = 5 - x^2 - y^2 = 5 - (x^2 + y^2)$ is the flipped paraboloid lifted 5 units up. The pictures are below:





The intersection of the surface $z = 5 - x^2 - y^2$ with the xy-plane is the intersection with the plane $z = 0$. The intersection is a curve on the xy-plane satisfying the equation:

$$0 = 5 - x^2 - y^2.$$

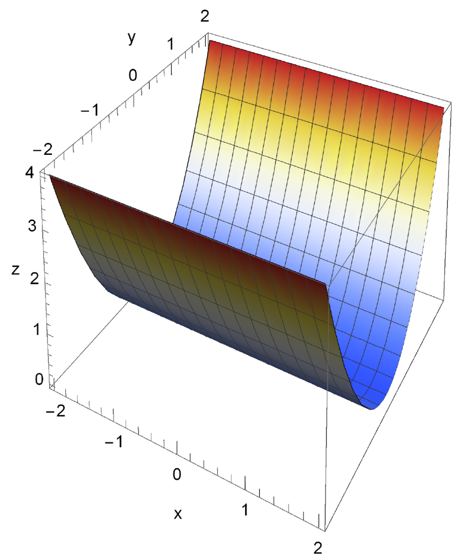
In other words,

$$x^2 + y^2 = 5.$$

The curve is the circle centered on the point $(0, 0)$ on the xy-plane with radius $\sqrt{5}$.

Occasionally, a function $z = f(x, y)$ of two variables is constant with respect to one variable. In practical terms, it means that the formula for $f(x, y)$ contains only one variable.

Example 4. Consider a function $z = f(x, y)$ where $f(x, y) = y^2$. The function $f(x, y)$ is constant with respect to x. The graph of the function looks as follows:



Of course: the cross-section of $z = y^2$ with the plane $x = a$ for any constant a is the parabola $z = y^2$ in the $x = a$ plane. So the graph is obtained by sliding the vertex of the parabola $z = y^2$ along the x-axis. Cross-sections with $y = b$ for any constant b are straight lines parallel to the x-axis clearly visible on the graph.

Important Surfaces That Are Not Graphs of Functions

We already saw surfaces in the xyz-space that are not graphs of functions $z = f(x, y)$. A **sphere** centered at a point (a, b, c) with radius r given by the equation:

$$(x - a)^2 + (y - b)^2 + (z - c)^2 = r^2$$

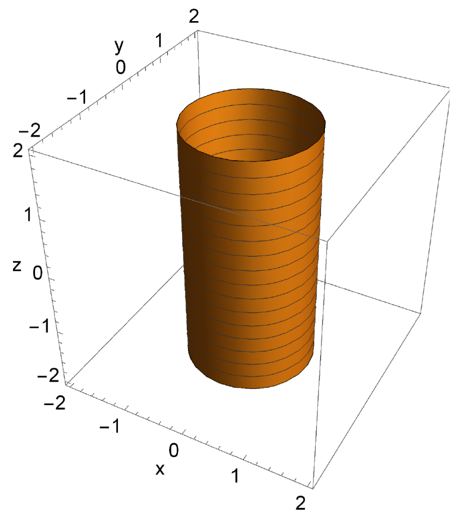
is a perfectly fine surface in the xyz-space but not a graph of a function $z = f(x, y)$ as it doesn't satisfy the vertical line test.

There are other important surfaces that are not graphs of functions.

Example 5. Consider a surface given by the xyz-equation:

$$x^2 + y^2 = 1.$$

The cross-section of the surface with any horizontal plane $z = c$ is the unit circle in that plane. So the surface is obtained by taking the unit circle on the xy-plane and sliding it along the z-axis.



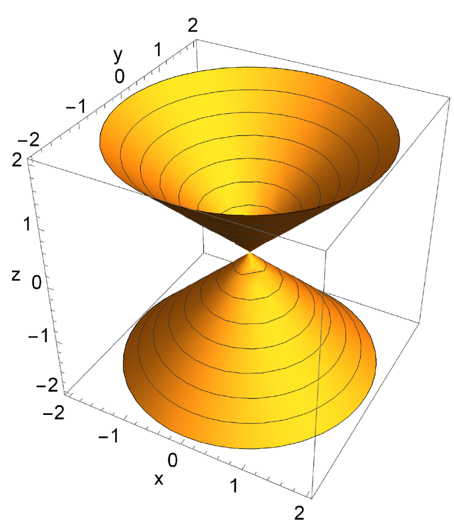
The surface is the infinite **cylinder** of radius 1 about the z-axis. The infinite **cylinder** of radius r about the z-axis has the equation:

$$x^2 + y^2 = r^2.$$

Example 6. Consider the surface:

$$z^2 = x^2 + y^2.$$

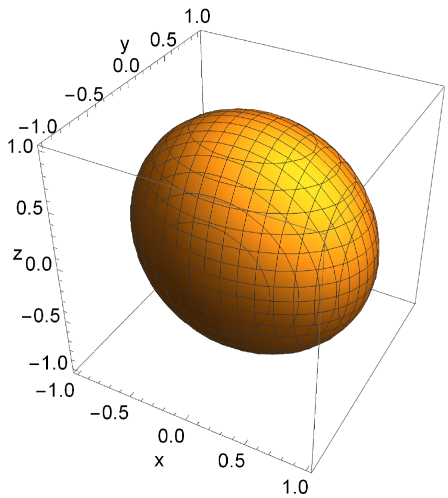
The cross-section of the surface with any horizontal plane $z = c$ is the circle in that plane with equation $c^2 = x^2 + y^2$. In other words, it is a circle in the plane $z = c$ centered at the point $(0, 0, c)$ with radius $|c|$. Like a cylinder, the surface is circularly symmetric abut the z-axis but the radii of the circles of intersection with $z = c$ are changing as z is changing. The radius of the circles on the plane $z = c$ is $|c|$. Hence, the surface is an infinite **cone** about the z-axis with the angle of $90°$ at the vertex.



Example 7. Consider the surface:

$$x^2 + 2y^2 + z^2 = 1.$$

The surface is almost a sphere centered at the origin with radius 1 except for the coefficient 2 at y. Just like for circles at ellipses, the coefficient squeezes the sphere in the y direction and produces an egg-like surface called an **ellipsoid**.



In this section, we built a library of useful surfaces in the xyz-space.