12.1 Functions of Two Variables — Key Points

In MTH 141 and MTH 142 we dealt with functions of one independent variable, typically denoted as:

$$y = f(x).$$

Multivariable calculus is the differentiable and integral calculus of functions of two, three and any number of independent variables. The standard notation for functions of two variables is:

$$z = f(x, y).$$

We have two independent variables x and y and the dependent variable z that depends on x and y. Functions of three variables are often denoted by:

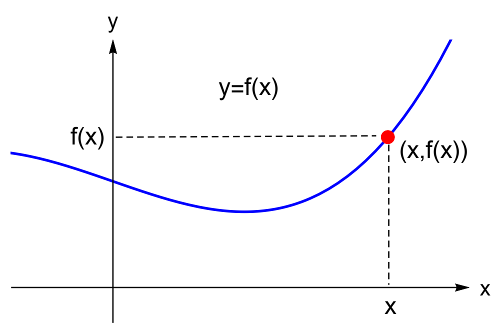
$$w = f(x, y, z).$$

We have three independent variables x, y and z and the dependent variable w.

We will study functions of two as well as three variables. Things get considerably more complicated for such functions. For example, the main way to visualize a function of one variable and interpret geometrically all concepts was through the graph of a function. What are graphs of functions of several variables?

Graphs of Functions of Two and Three Variables

Recall that the graph of a single-variable function $y = f(x)$ is a curve of the xy-plane — the curve that consists of all **pairs** of the form: $(x, f(x))$ where x is the domain of f:



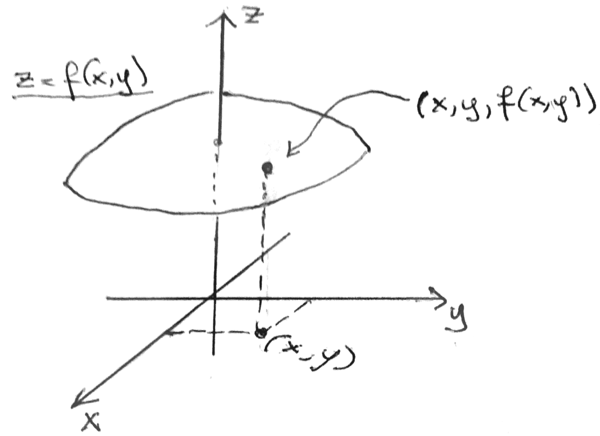
The domain of f — the collection of all inputs x for which the function is defined — can be visualized as a subset of the x-axis.

What is the graph of a function of two variables $z = f(x, y)$? Following the same logic as in the single-variable case, the graph of $z = f(x, y)$ is the collection of all **triples**:

$$(x, y, f(x, y)).$$

That is, a collection of triples of the form: a pair (x, y) in the domain of f and the value of the function $f(x, y)$ for the pair (x, y). A set of triples is located in the 3D, three dimensional,

xyz-space. It is the set of triples (x, y, z) that satisfy the equation $z = f(x, y)$. In general, such a collection of triples forms a **surface** in the xyz-space:



The domain of f is the collection of pairs (x, y) for which $f(x, y)$ is defined. The domain can be visualized as a region on the xy-plane in the xyz-space.

We will use graphs of functions of two variables although they are considerably harder to visualize than curves on the plane.

How about functions of three variables, $w = f(x, y, z)$? What is the graph of such a function? Following our logic, and the established mathematical terminology, the graph of $w = f(x, y, z)$ is the collection of quadruples $(x, y, z, f(x, y, z))$ corresponding to all triples (x, y, z) in the domain of f. The domain is a region in the xyz-space; the graph is a subset of the 4D, four dimensional, $xyzw$-space. We can define the graph this way but obviously we cannot visualize it.

Numerical Representation of Functions of Two Variables

A numerical representation of a function $y = f(x)$ — a table with two rows — is very simple and quickly gives you insight into the behavior of the function if not a formula:

x	0	1	2	3	4	5	6
$f(x)$	0	1	4	9	16	25	36

A function $z = f(x, y)$ has two independent variables, x and y, so its table of value is a rectangular array. Let's look at an example.

Example 1. The body mass index (BMI) is a value that attempts to quantify a person's body fat based on their height h and weight w. In functional notation, we write:

$$I = I(h, w)$$

where h is a person's height, in inches, w is weight in pounds. Here is the table for $w = 120, 140, 160, 180, 200$ and $h = 60, 63, 66, 69, 72, 75$. The table must contain outputs I for all corresponding pairs (h, w) so it has 30 entries:

	Weight w (lbs)				
Height h (inches)	120	140	160	180	200
60	23.4	27.3	31.2	35.2	39.1
63	21.3	24.8	28.3	31.9	35.4
66	19.4	22.6	25.8	29.0	32.3
69	17.7	20.7	23.6	26.6	29.5
72	16.3	19.0	21.7	24.4	27.1
75	15.0	17.5	20.0	22.5	25.0

We can read the values from the table by choosing a row and a column. For example:

$$f(66, 140) = 22.6.$$

The body mass index of a person who weighs 140 lbs and is 66 inches tall is 22.6. According to the CDC chart[1], it is normal, healthy weight.

Let's observe that it is not easy to tell much about the behavior of the function $I = I(h, w)$ based on its table of values. Neither can we guess a formula for the function.

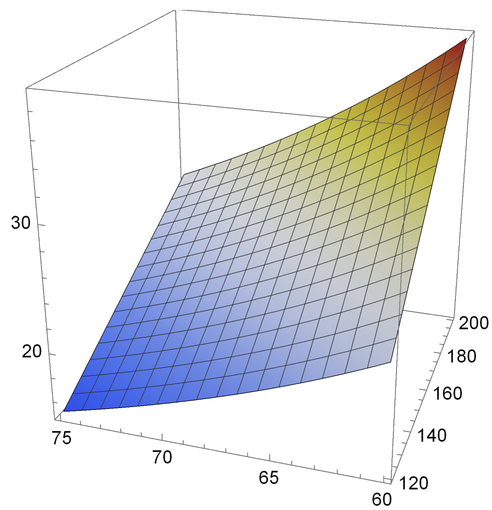
The formula for the body mass index is:

$$I(h, w) = \frac{w \cdot 703}{h^2}$$

(The values in the table are rounded off to one decimal place.)

Curiously, a table of values like the one above are used by graphing software packages like Mathematica to draw graphs of functions of two variables. Below is the graph of $I = I(h, w)$ over the rectangle $60 \leq h \leq 75$, $120 \leq w \leq 200$ on the hw-plane:

[1] https://www.cdc.gov/healthyweight/assessing/bmi/adult_bmi/index.html, accessed: 6/26/20

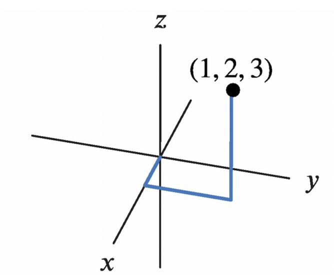


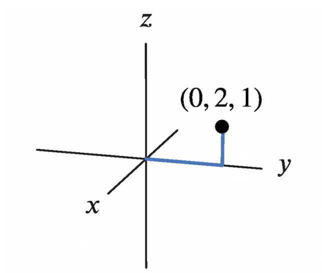
A Tour of xyz-Space

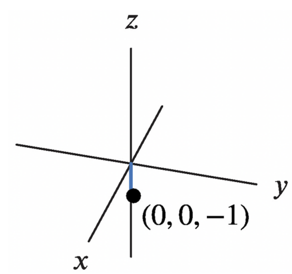
Let's review basics about 3D space.

Example 2. Describe the position in the xyz-space of the points with coordinates $(1, 2, 3)$, $(0, 2, 1)$, and $(0, 0, -1)$.

Solution. Here are the pictures illustrating the position of the points:





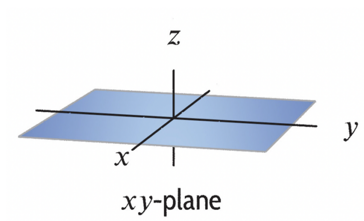


Let's look at some simple surfaces given by xyz-equations.

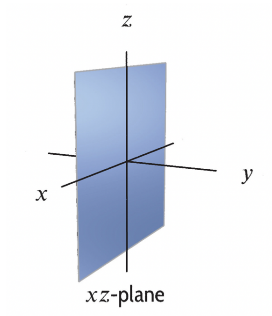
Example 3. What are the surfaces given by the equations:

(a) $z = 0$ (b) $y = 0$ (c) $x = 0$

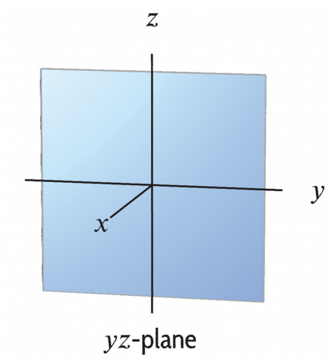
Solution. (a) $z = 0$ is the so-called xy-plane:



(b) $y = 0$ is the so-called xz-plane:



(c) $x = 0$ is the so-called yz-plane:

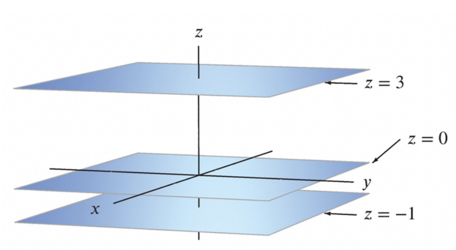


The three planes are called the **coordinate planes**.

Example 4. What are the surfaces given by the equations:

(a) $z = -1$ (b) $z = 3$

Solution. Both surfaces are horizontal planes, parallel to the xy-coordinate plane:

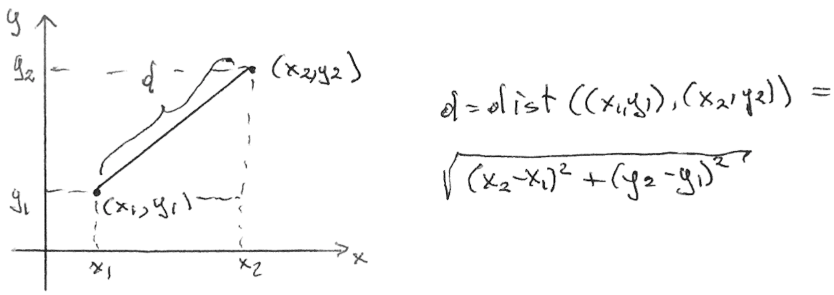


Example 5. Which of the points $A = (1, -1, 0)$, $B = (0, 3, 4)$, $C = (2, 2, 1)$, and $D = (0, -4, 0)$ lies closest to the xz-plane? Which point lies on the y-axis?

Solution. The magnitude of the y-coordinate gives the distance to the xz-plane. Point A lies closest to that plane, because it has the smallest y-coordinate in magnitude. To get to a point on the y-axis, we move along the y-axis, but we don't move at all in the x- or the z-direction. Thus, a point on the y-axis has both its x- and z-coordinates equal to zero. The only point of the four that satisfies this is D.

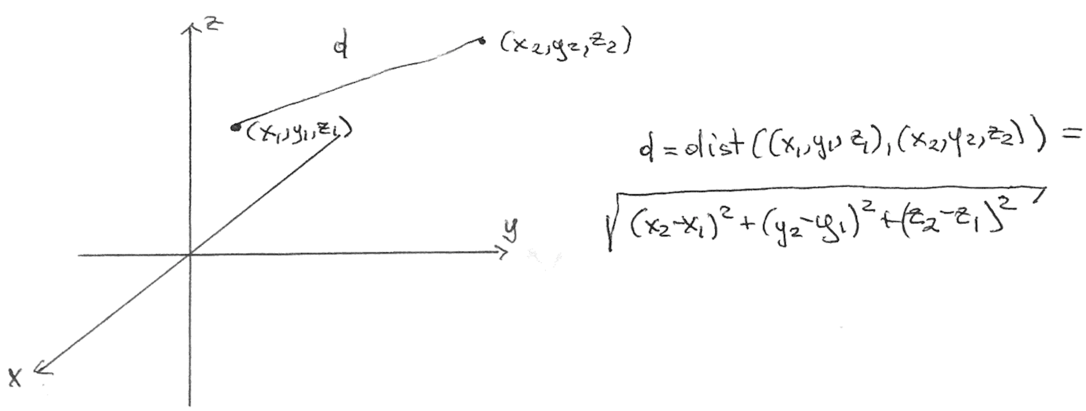
The Distance in 3D Space

Recall that the distance between two points on the xy-plane is:



The proof is very easy via the Pythagorean Theorem.

A formula for the distance between two points (x_1, y_1, z_1) and (x_2, y_2, z_2) in the xyz-space can be easily derived by applying the the Pythagorean Theorem twice:



Distance in 3D Space

The distance between points (x_1, y_1, z_1) and (x_2, y_2, z_2) is:

$$\text{dist}((x_1, y_1, z_1), (x_2, y_2, z_2)) = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

Example 6. Find the distance between $A = (1, 2, 1)$ and $B = (-3, 1, 2)$.

Solution. We use the distance formula:

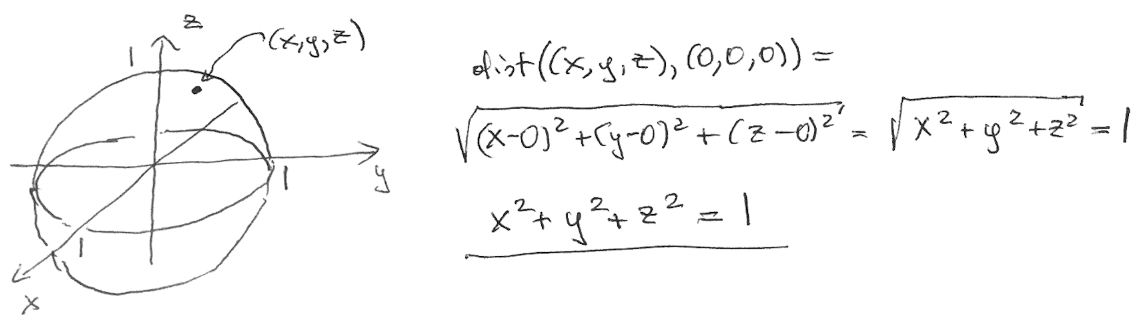
$$\text{dist}((1, 2, 1), (-3, 1, 2)) = \sqrt{(-3 - 1)^2 + (1 - 2)^2 + (2 - 1)^2} = \sqrt{18} \approx 4.243.$$

Spheres in xyz-Space

Having a distance formula, we can find equations of spheres.

Example 7. Find an equation of the **unit sphere**; that is, the sphere of radius 1 centered at the origin.

Solution. The unit sphere is the collection of all points (x, y, z) whose distance from the origin — $(0, 0, 0)$ — is 1:



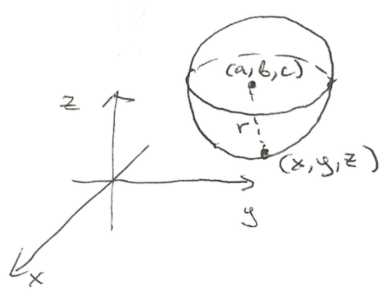
As illustrated by the picture, we obtain the equation:

$$\sqrt{x^2 + y^2 + z^2} = 1$$

which is traditionally written as:

$$x^2 + y^2 + z^2 = 1$$

Using the distance formula, we can easily get an equation of any sphere centered at a point (a, b, c) with radius r:



$$(x-a)^2+(y-b)^2+(z-c)^2=r^2$$

Equation of a Sphere

The sphere centered at a point (a, b, c) with radius r is given by the equation:

$$(x-a)^2+(y-b)^2+(z-c)^2=r^2.$$