March 24-26

16.2 Iterated Integrals

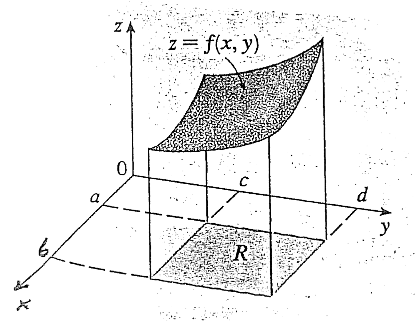
Last time you learned the definition and the meaning of the double integral of a given function $f(x,y)$ over a given region R in the xy-plane:

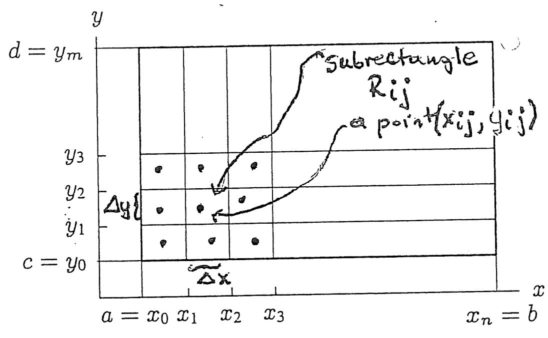
$$\int_R f(x,y)\, dA .$$

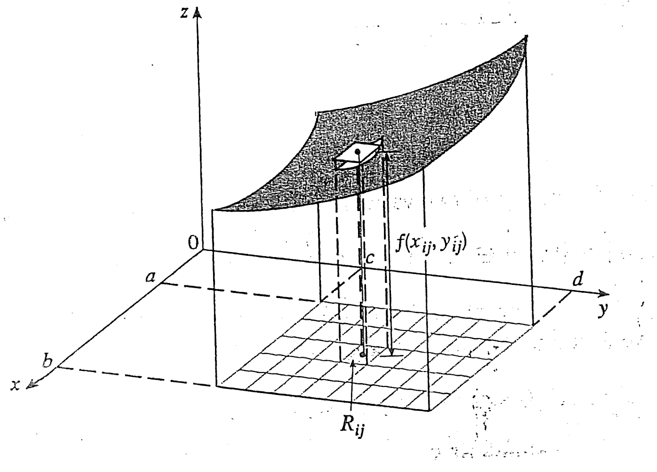
How is such integral defined? In terms of Riemann sums. Assume for simplicity that R is a rectangle:

$$R: \ a \le x \le b \ , \ c \le y \le d .$$

(The definition is the same for an arbitrary region R.) The pictures should refresh your memory about the definition of Riemann sums and the double integral:







$$\int_R f(x,y)\,dx\,dy = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0}} \underbrace{\sum f(x_{ij}, y_{ij})\,\Delta x\,\Delta y}_{\text{a Riemann sum}}$$

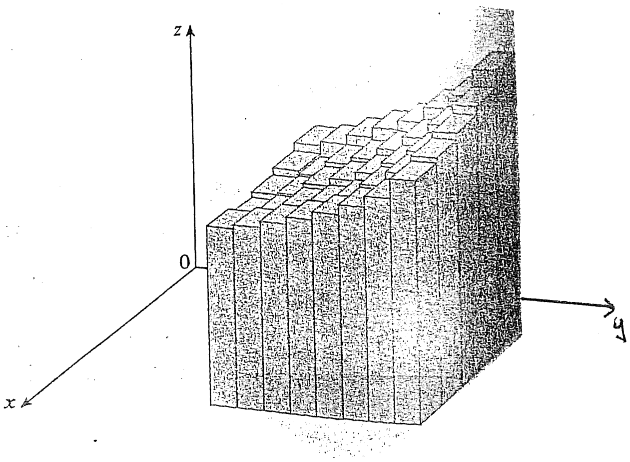
Also denoted:

$$\int_R f(x,y)\, dA = \int_R f(x,y)\, dx\, dy = \int_R f(x,y)\, dy\, dx.$$

The element of area.

Philosophically: $\int_R f(x,y)\,dA$ is "the value of the function f" times the area of R. But the value of f changes over R, so you have to subdivide R etc. etc.

If $f(x,y) \geq 0$ in R, then $\int_R f(x,y)\,dx\,dy$ is the volume under the graph:



Riemann sums approximate the volume. The approximation gets better and better as $\Delta x \to 0$, $\Delta y \to 0$.

Riemann sums are not very handy for evaluating double integrals. Iterated integrals are.

Calculating Double Integrals Using Iterated Integrals

Let R be a rectangle $a \le x \le b$, $c \le y \le d$. Let $f(x,y)$ be continuous on R. Then

$$\int_R f(x,y)\,dA = \int_c^d \Big(\int_a^b \overset{\substack{y \text{ fixed} \\ \downarrow}}{f(x,y)\,dx} \Big)\, dy = \int_a^b \Big(\int_c^d \overset{\substack{x \text{ fixed} \\ \downarrow}}{f(x,y)\,dy} \Big)\, dx$$

interated integrals

Ex: $f(x,y) = x^2 y$, R: $0 \le x \le 1$, $0 \le y \le 2$.

Evaluate $\int_R f(x,y)\,dA$.

$$\int_R (x^2 y)\,dA = \int_0^2 \int_0^1 (x^2 y)\,dx\,dy = \int_0^1 \int_0^2 (x^2 y)\,dy\,dx =$$

Integrate with respect to x first

Integrate with respect to y first. Let's do this one.

$$= \int_0^1 2x^2\,dx = \frac{2}{3}x^3 \Big|_0^1 = \underline{\underline{\frac{2}{3}}}$$

Inside integral:

$$\int_0^2 (\underset{\substack{\uparrow \\ \text{constant}}}{x^2} y)\,dy = \left(x^2 \frac{1}{2} y^2\right) \Big|_{\underset{y=0}{0}}^{\overset{y=2}{2}} = 2x^2$$

We could use the other iterated integral. For rectangular regions it usually doesn't matter.

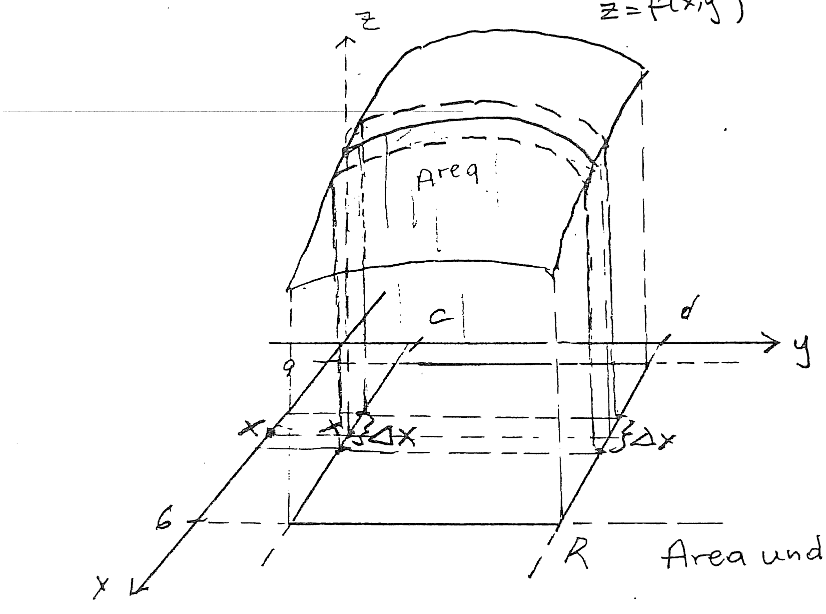
limits for x

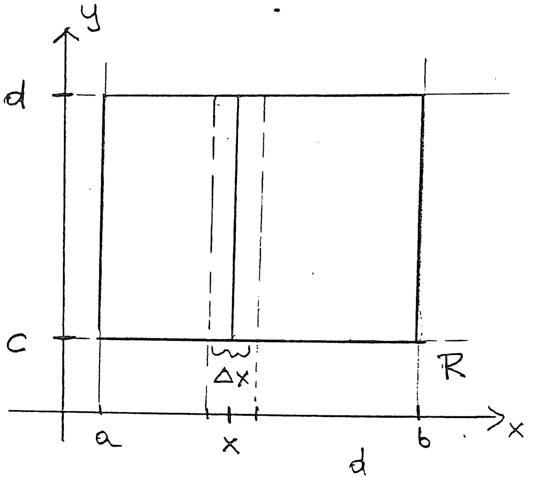
$$\int_0^2 \int_0^1 (x^2 y)\,dx\,dy = \int_0^2 \frac{1}{3} y\,dy = \frac{1}{6} y^2 \Big|_0^2 = \frac{4}{6} = \frac{2}{3}$$

limits for y

$$\int_0^1 (x^2 \underset{\uparrow \text{ constant}}{y})\,dx = \frac{1}{3} x^3 y \Big|_{x=0}^{x=1} = \frac{1}{3} y$$

Why do interated integrals work? We will justify why in the case when $f(x,y) \geq 0$ over R and R is a rectangle; In that case, $\int_R f\,dA$ is the volume under the graph. In the definition of the double integral, we calculated the volume by dividing into subrectangles and thin vertical pillars. The volume can be computed by dividing into slabs:





Area under the cross-section $f(x,y) = \int_c^d f(x,y)\,dy$

With x-fixed:

Volume above the x-slab $\cong$ Area $\cdot\, \Delta x \cong \left(\int_c^d f(x,y)\,dy\right) \cdot \Delta x$

Total volume $= \lim\limits_{\Delta x \to 0} \sum\limits_{\substack{\text{over} \\ x\text{-slabs}}} \left(\int_c^d f(x,y)\,dy\right) \cdot \Delta x = \int_a^b \left(\int_c^d f(x,y)\,dy\right) dx$

Iterated integrals can be used to calculate double integrals over non-rectangular regions. The interpretation in terms of slabs is very helpful in that case.

Ex: Find

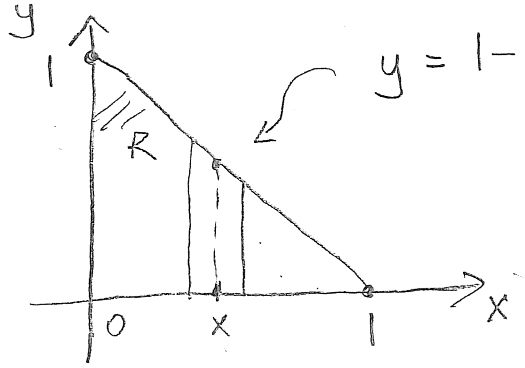
$$\int_R (x+2y)\,dA$$

where R is the triangle with vertices $(0,0)$, $(1,0)$, $(0,1)$.

Step 1: Sketch the region

Step 2: Set up an iterated integral

Step 3: Evaluate the iterated integral.



The key is to properly setup an iterated integral and the limits of the inside integral which in non-rectangular case will not be constant.

Suppose we calculate integral by cutting the volume into x-slabs and then adding the volumes of slabs. What is the volume of each x-slab?

$$\text{Volume of x-slab} = \left(\int_0^{1-x} (x+2y)\,dy\right)\cdot \Delta x$$

$$\text{Total volume} = \int_R (x+2y)\,dA = \lim_{\Delta x\to 0} \sum_{\substack{\text{over}\\ \text{slabs}}} \left(\int_0^{1-x} (x+2y)\,dy\right)\Delta x =$$

$$= \int_0^1\left(\int_0^{1-x} (x+2y)\,dy\right)dx$$

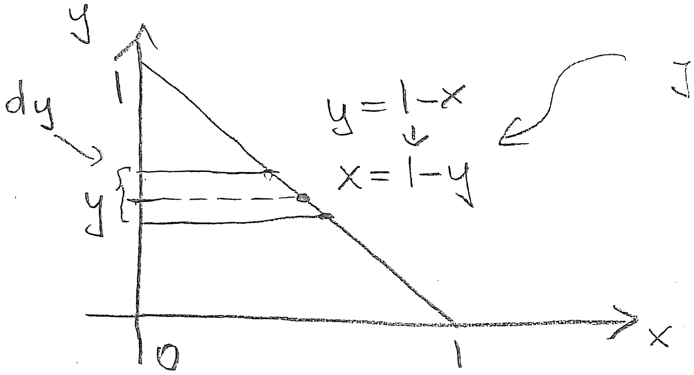
$$\int_0^1 \left(\int_0^{1-x} (x+2y)\,dy \right) dx =$$

$$= \int_0^1 \left(x(1-x) + (1-x)^2\right) dx = \int_0^1 (1-x)(x+1-x)\,dx = \underline{\underline{\tfrac{1}{2}}}$$

$$\int_0^{1-x} (x+2y)\,dy = (xy+y^2)\Big|_{y=0}^{y=1-x} =$$

$$= x(1-x) + (1-x)^2$$

What if we cut into y-slabs?



Inside integration with respect to x. Inside limits may depend on y.

$$\int_0^1 \left(\int_0^{1-y} (x+2y)\,dx \right) dy = \int_0^1 \left(\left(\tfrac{1}{2}x^2 + 2yx\right)\Big|_{\underset{x}{\overset{\shortparallel}{0}}}^{\overset{x}{\overset{\shortparallel}{1-y}}} \right) dy =$$

$$= \int_0^1 \left(\tfrac{1}{2}(1-y)^2 + 2(1-y)y\right) dy = \frac{1}{2}.$$

In this example it didn't matter which iterated integral came first. Sometimes it does matter.

Ex: Consider the iterated integral:

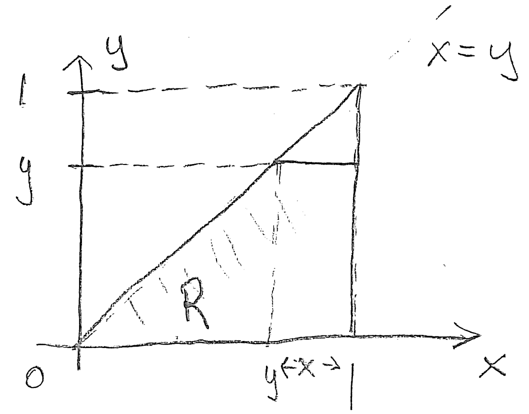
$$\int_0^1 \int_y^1 e^{x^2}\,dx\,dy$$

(a) Sketch the region of R for the corresponding double integral.

(b) Evaluate the integral.

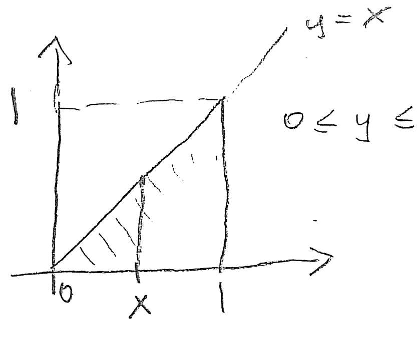
$0 \le y \le 1 \qquad y \le x \le 1$

R:



R: Triangle (0,0), (1,1), (1,0).

$$\int_0^1 \int_y^1 e^{x^2}\,dx\,dy = \int_R e^{x^2}\,dA$$



$\int_y^1 e^{x^2}\,dx$ cannot be found.

Change order of integration: $\int_R e^{x^2}\,dA = \int_0^1 \int_?^? e^{x^2}\,dy\,dx =$

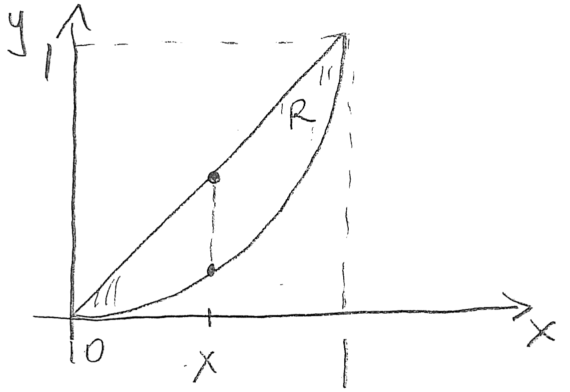
$$= \int_0^1 \left(\int_0^x e^{x^2}\,dy\right)dx = \int_0^1 \left(e^{x^2} y \Big|_{y=0}^{y=x}\right)dx = \int_0^1 x e^{x^2}\,dx =$$

$$= \frac{1}{2} e^{x^2}\Big|_0^1 = \underline{\frac{e}{2} - \frac{1}{2}}$$

Ex: Consider a metal plate R bounded by $y=x$ and $y=x^2$ with density of mass:

$$\delta(x,y) = 1+xy \ \frac{kg}{m^2}$$

where x, y are in meters. Find the total mass of the plate.



$$M = \int_R \delta(x,y)\, dA =$$

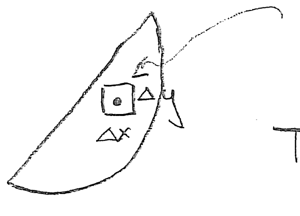
$$= \int_0^1 \left(\int_{x^2}^{x} (1+xy)\, dy \right) dx =$$

$$= \int_0^1 \left(x + \tfrac{1}{2}x^3 - x^2 - \tfrac{1}{2}x^5\right) dx = \left(\tfrac{1}{2}x^2 + \tfrac{1}{8}x^4 - \tfrac{1}{3}x^3 - \tfrac{1}{12}x^6\right)\Big|_0^1 = \tfrac{5}{24}\ kg$$

$$\int_{x^2}^{x} (1+xy)\, dy = \left(y + \tfrac{1}{2}xy^2\right)\Big|_{y=x^2}^{y=x} = \left(x + \tfrac{1}{2}x^3\right) - \left(x^2 + \tfrac{1}{2}x^5\right) =$$

$$= x + \tfrac{1}{2}x^3 - x^2 - \tfrac{1}{2}x^5$$

Why is M equal to the double integral?

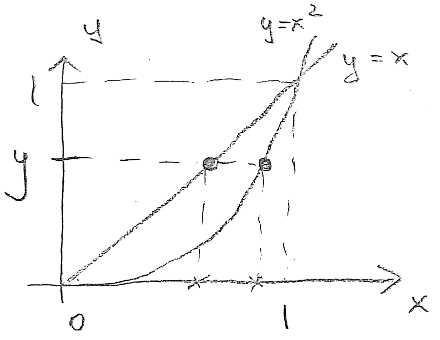


Mass of the Δx by Δy piece $\cong \delta(x,y) \cdot \Delta x \cdot \Delta y$

$$\text{Total mass} \cong \sum_{\substack{\text{over}\\ \text{all pieces}}} \delta(x,y)\,\Delta x \Delta y \ \xrightarrow[\substack{\Delta x \to 0\\ \Delta y \to 0}]{} \ \int_R \delta(x,y)\, dA$$

$$M = \int_R \delta(x,y)\, dA$$

How would the other iterated integral look?

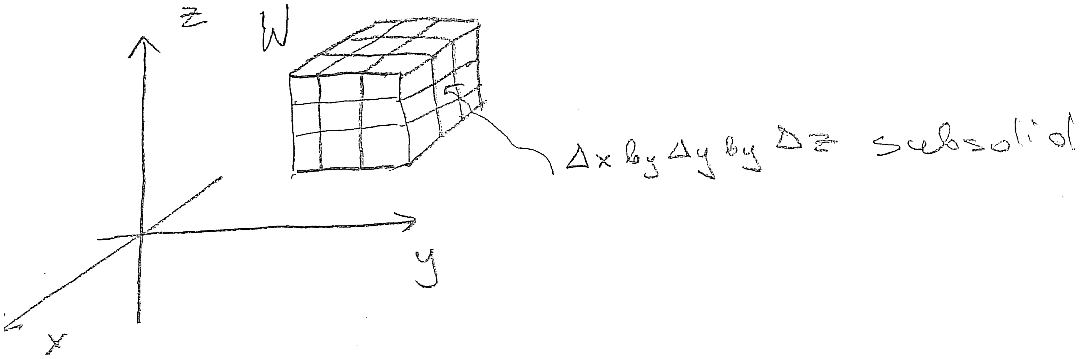


$$\int_0^1 \left(\int_y^{\sqrt{y}} \delta(x,y)\,dx \right) dy = \frac{5}{24}\text{ kg}$$

Triple Integrals

You can guess what they are. Let $f(x,y,z)$ be given and let a solid W in the xyz-space be given. For example, let W be a simple rectangular solid:

$$W: \quad a \le x \le b, \quad c \le y \le d, \quad p \le z \le q$$



We define the triple integral as:

$$\int_W f(x,y,z)\,dV = \lim_{\substack{\Delta x \to 0 \\ \Delta y \to 0 \\ \Delta z \to 0}} \sum_{\substack{\text{all} \\ \text{subsolids}}} f(x_{i,j,k}, y_{i,j,k}, z_{i,j,k})\,\Delta x \Delta y \Delta z$$

The integral is the limit of Riemann sums obtained by dividing W into small subsolids Δx by Δy by Δz, choosing a point $(x_{i,j,k}, y_{i,j,k}, z_{i,j,k})$ from each subsolid and multiplying the value of f at that point by the volume of the subsolid.

Philosophically, $\int_W f(x,y,z)\,dV$ is "the value of f times the volume of W", but if f changes over W, we have to divide W into small pieces etc....

Clearly from the definition:

$$\int_W 1\, dV = \text{volume}(W).$$

Similarly, as

$$\int_R 1\, dA = \text{Area}(R).$$

Also similarly as for double integrals, we can find triple integrals via iterated integrals:

$$\int_W f(x,y,z)\,dV = \int_p^q \left(\int_c^d \left(\int_a^b f(x,y,z)\,dx \right) dy \right) dz$$

Also for non-rectangular solids.

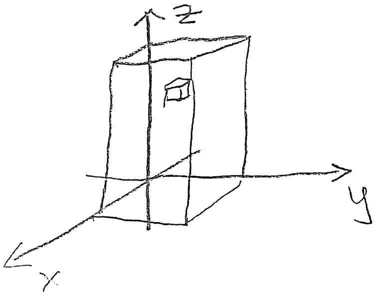
Ex: Consider a metal solid W in the xyz-space

W: $0 \le x \le 1$, $0 \le y \le 1$, $0 \le z \le 2$

where x, y, z are in centimeters. The density of mass in W is given by:

$$\delta(x,y,z) = (x+y+z+1)\ \frac{g}{cm^3}$$

Find the total mass of W.



If the density was constant, say

$\delta = 3\ g/cm^3$

what the total mass would be?

$M = \text{Vol}(W) \cdot \delta = 2\,cm^3 \cdot 3\ g/cm^3 = 6g.$

But the density is not constant so we have to divide the solid into small subsolids over which $\delta(x,y,z)$ change only a little, estimate the mass of each subsolid:

$$\text{Mass of subsolid} \cong \overset{\text{a point from the subsolid}}{\delta(x,y,z)} \cdot \Delta x \Delta y \Delta z$$

$$M = \text{Total mass} \cong \sum_{\substack{\text{over} \\ \text{all subsolids}}} \delta(x,y,z)\,\Delta x\,\Delta y\,\Delta z$$

In other words, we have to take a triple integral:

$$M = \int_W \delta(x,y,z)\,dV.$$

We evaluate the integral using iterated integrals:

$$M = \int_0^2 \int_0^1 \underbrace{\int_0^1 (x+y+z+1)\, dx}\, dy\, dz =$$

$$= \int_0^2 \int_0^1 \left(\left(\tfrac{1}{2}x^2 + yx + zx + x\right)\Big|_{x=0}^{x=1} \right) dy\, dz =$$

$$= \int_0^2 \int_0^1 \left(\tfrac{1}{2} + y + z + 1\right) dy\, dz = \int_0^2 \underbrace{\int_0^1 \left(\tfrac{3}{2} + y + z\right) dy}\, dz =$$

$$= \int_0^2 \left(\left(\tfrac{3}{2}y + \tfrac{1}{2}y^2 + zy\right)\Big|_{y=0}^{y=1} \right) dz =$$

$$= \int_0^2 \left(\tfrac{3}{2} + \tfrac{1}{2} + z\right) dz = \left(2z + \tfrac{1}{2}z^2\right)\Big|_0^2 = \underline{\underline{6\ g}}$$

After calculating each integral, one variable disappears.

Iterated integrals work for non-rectangular solids. You have to be careful about the limits.

Ex: Let $f(x,y,z) = x+y$. Find

$$\int_W f \, dV$$

where W is the solid bounded by the xy-plane, yz-plane, xz-plane, and the plane

$$P: \quad \frac{x}{3} + \frac{y}{2} + \frac{z}{6} = 1.$$

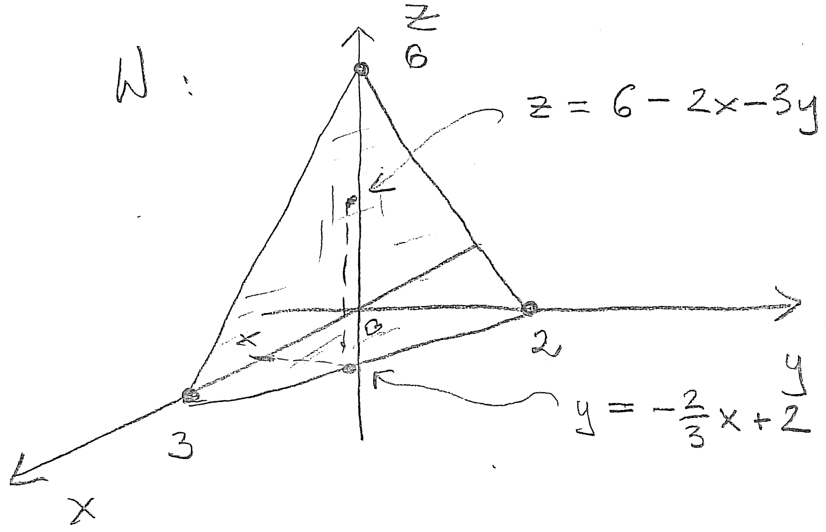
$z = 0$:

$\frac{x}{3} + \frac{y}{2} = 1 \rightarrow y = -\frac{2}{3}x + 2$

$z = 6 - 2x - 3y$

Let's sketch the solid so we can set up an iterated integral.

W :



Normal to P:

$\vec{n}_P = \langle \frac{1}{3}, \frac{1}{2}, \frac{1}{6} \rangle$

P intersects the

x-axis at (3, 0, 0)

y-axis at (0, 2, 0)

z-axis at (0, 0, 6)

W is the pyramid in the first octant.

$$\int_W f \, dV = \int_0^3 \int_0^{-\frac{2}{3}x+2} \int_0^{6-2x-3y} (x+y) \, dz \, dy \, dx$$

$$\int_0^3 \left(\int_0^{-\frac{2}{3}x+2} \left(\underbrace{\int_0^{6-2x-3y} (x+y)\, dz \right)} dy \right) dx =$$

$$= \int_0^3 \left(\int_0^{-\frac{2}{3}x+2} (x+y)(6-2x-3y)\, dy \right) dx =$$

—

$$(x+y)(6-2x-3y) = 6x-2x^2-3xy+6y-2xy-3y^2 =$$

$$= 6x-2x^2-3y^2+6y-5xy$$

—

$$= \int_0^3 \underbrace{\left(\int_0^{-\frac{2}{3}x+2} (6x-2x^2-3y^2+6y-5xy)\, dy \right)} dx =$$

$$= \int_0^3 \underbrace{\left(\left(6xy-2x^2y-y^3+3y^2-\frac{5}{2}xy^2 \right) \Big|_{y=0}^{y=-\frac{2}{3}x+2} \right)} dx =$$

↙

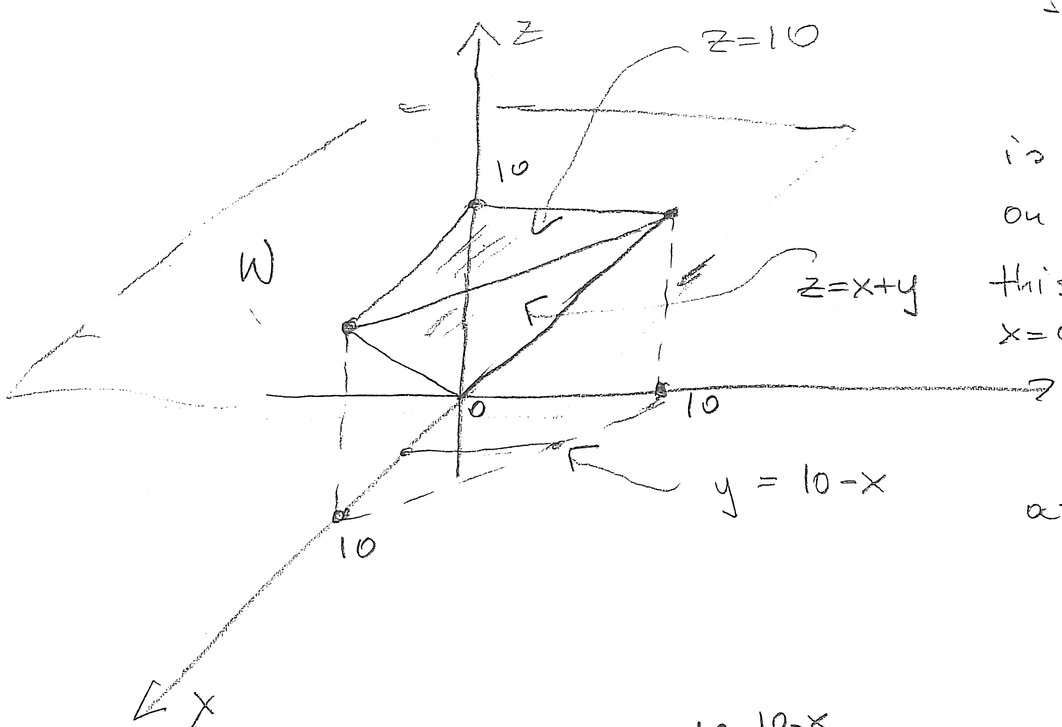
$$6x\left(-\frac{2}{3}x+2\right) - 2x^2\left(-\frac{2}{3}x+2\right) - \left(-\frac{2}{3}x+2\right)^3 + 3\left(-\frac{2}{3}x+2\right)^2 -$$

$$- \frac{5}{2}x\left(-\frac{2}{3}x+2\right)^2 = \underline{\frac{14}{27}x^3 - \frac{8}{3}x^2 + 2x + 4}$$

↙

$$= \int_0^3 \left(\frac{14}{27}x^3 - \frac{8}{3}x^2 + 2x + 4 \right) dx = \underline{\underline{\frac{15}{2}}}$$

Ex: Find the volume of the solid bounded by $z = x + y$, $z = 10$, and the planes $x = 0$, $y = 0$.



Intersection of $z = x+y$ with $z=10$ is the $10 = x+y$ line on the $z = 10$ plane. this line intersects $x=0$ plane at $(0, 10, 10)$ and $y=0$ plane at $(10, 0, 10)$.

$$\text{Vol}(W) = \int_W 1 \, dV = \int_0^{10} \int_0^{10-x} \int_{x+y}^{10} 1 \quad dz\, dy\, dx =$$

$$= \int_0^{10} \int_0^{10-x} \left(z \Big|_{z=x+y}^{z=10} \right) dy\, dx =$$

$$= \int_0^{10} \left(\int_0^{10-x} (10 - x - y)\, dy \right) dx = \int_0^{10} \left(\left(10y - xy - \tfrac{1}{2} y^2 \Big|_{y=0}^{y=10-x} \right) dx = \right.$$

$$= \int_0^{10} \left(10(10-x) - x(10-x) - \tfrac{1}{2}(10-x)^2 \right) dx = \int_0^{10} \left(50 - 10x + \tfrac{1}{2}x^2 \right) dx = \frac{500}{3}$$

$$(10-x)(10-x-5+\tfrac{1}{2}x) = (10-x)(5-\tfrac{1}{2}x) =$$

$$= 50 - 5x - 5x + \tfrac{1}{2}x^2 = 50 - 10x + \tfrac{1}{2}x^2$$

Ex: Sketch the solid of integration W corresponding to the iterated integral:

$$\int_0^1 \int_{-\sqrt{1-z^2}}^{\sqrt{1-z^2}} \int_{-\sqrt{1-y^2-z^2}}^{\sqrt{1-y^2-z^2}} f(x,y,z)\,dx\,dy\,dz$$

W upper half of the unit ball