14.1, 14.2 Cont'd

Let $z = f(x,y)$. We defined partial derivatives functions denoted:

$$f_x(x,y) = \frac{\partial z}{\partial x} = \frac{\partial}{\partial x}[f(x,y)] = z_x = \frac{\partial f}{\partial x} = f_x$$

$$f_y(x,y) = \frac{\partial z}{\partial y} = \frac{\partial}{\partial y}[f(x,y)] = z_y = \frac{\partial f}{\partial y} = f_y$$

Computing partials algebraically is easy: we consider one variable to be a constant and differentiate with respect to the other.

Ex: Let $f(x,y) = x^3 + 3x^2y + y^2$. Find $f_x(x,y)$ and $f_y(x,y)$.

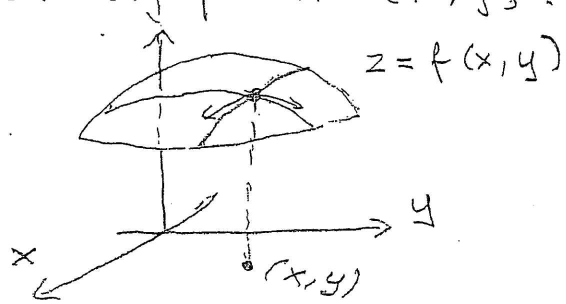
$$f_x(x,y) = \frac{\partial}{\partial x}[x^3 + 3x^2y + y^2] = 3x^2 + 6xy$$

y constant,
take the derivative in x.

$$f_y(x,y) = \frac{\partial}{\partial y}[x^3 + 3x^2y + y^2] = 3x^2 + 2y$$

x constant,
take the derivative in y.

Geometrically, we found slopes of $z = f(x,y)$ in the x and y directions at any point (x,y):



Ex: $f(t,s) = t^2 e^{t \cdot s}$

$$\frac{\partial f}{\partial t} = \frac{\partial}{\partial t}\left[t^2 e^{t \cdot s} \right] = 2te^{ts} + t^2 s e^{ts}$$

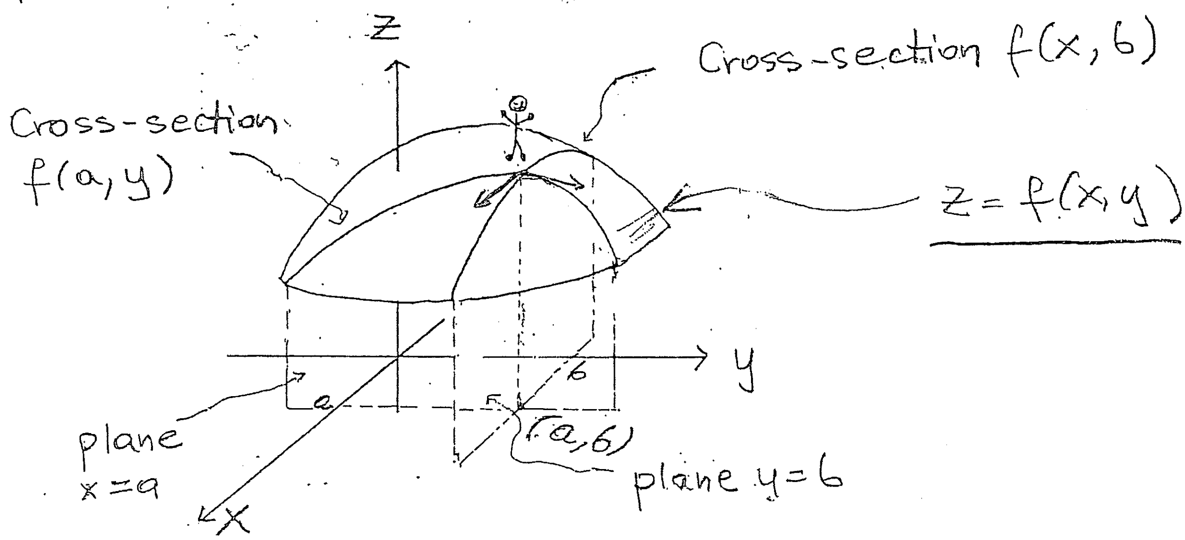
↑ s constant

$$\frac{\partial f}{\partial s} = \frac{\partial}{\partial s}\left[t^2 e^{ts} \right] = t^2 \cdot t e^{ts} = t^3 e^{ts}$$

↑ t constant

Practice!

Since partial derivatives are ordinary derivatives of cross-sections, they can be interpreted as rates of change.



$$f_x(a,b) = \frac{d}{dx}\Big|_{x=a}\left[f(x,b) \right] = \lim_{\Delta x \to 0} \frac{f(a+\Delta x, b) - f(a,b)}{\Delta x}$$

$f_x(a,b)$ $\quad \frac{\text{units of } z}{\text{unit of } x}$ - rate of change of z with respect to x at $x=a$ with y fixed at $y=b$.

since

$$f_x(a,b) = \lim_{\Delta x \to 0} \frac{f(a+\Delta x, b) - f(a,b)}{\Delta x}$$

we can approximate:

(*) $f_x(a,b) \cong \dfrac{f(a+\Delta x, b) - f(a,b)}{\Delta x}$ for Δx "small".

Often used if we have a contour diagram or a table of values for $f(x,y)$.

All goes the same for $f_y(a,b)$:

$$f_y(a,b) = \frac{d}{dy}\Big|_{y=b} [f(a,y)] = \lim_{\Delta y \to 0} \frac{f(a, b+\Delta y) - f(a,b)}{\Delta y}$$

$f_y(a,b)$ $\dfrac{\text{units of } z}{\text{unit of } y}$ - rate of change of z with respect to y at $y=b$ with $x=a$ fixed.

(*) $f_y(a,b) \cong \dfrac{f(a, b+\Delta y) - f(a,b)}{\Delta y}$ for Δy "small".

The approximation forumulas (*) give:

$f(a+\Delta x, b) \cong f(a,b) + f_x(a,b) \cdot \Delta x$ for Δx "small"

$f(a, b+\Delta y) \cong f(a,b) + f_y(a,b) \cdot \Delta y$ for Δy "small"

All as in MTH 141 as one variable is fixed.

Ex. Suppose that your weight, w, in pounds, is a function $w = f(c, m)$ of the number of calories, c, you consume daily and the number of minutes, m, you exercise daily.

(a) What are the units of $\frac{\partial w}{\partial c}(c, m)$?

(b) What is the practical meaning of the statement $\frac{\partial w}{\partial c}(2100, 20) = 0.007$?

(c) What are the units of $\frac{\partial w}{\partial m}(c, m)$?

(d) What is the practical meaning of the statement $\frac{\partial w}{\partial m}(2100, 20) = -0.2$?

(e) Assume that $w(2100, 20) = 120$, $\frac{\partial w}{\partial m}(2100, 20) = -0.2$. Estimate $w(2100, 25)$.

$$\underset{lb}{w} = f(\underset{cal}{c}, \underset{min}{m})$$

(a) $\frac{\partial w}{\partial c}(c, m)$ $\quad \frac{lb}{cal}$

(With minutes of excercise fixed, how fast is your weight increasing when number of calories increases.)

(b) $\frac{\partial w}{\partial c}(2100, 20) = 0.007 \, \frac{lb}{cal}$

When you excercise 20 min daily and consume 2100 calories daily, your weight is increasing at the (instantaneous) rate of $0.007 \frac{lb}{cal}$ if you increase your calories (and keep 20 min constant).

(c) $\frac{\partial w}{\partial m}(c, m)$ $\quad \frac{lb}{min}$

(With calories fixed how fast is your weight decreasing as you increase minutes of excercise.)

(d) $\frac{\partial w}{\partial m}(2100, 20) = -0.2 \frac{lb}{min}$

When you excercise 20 min daily and cousume 2100 calories your weight is changing at the (instantaneous) rate of $-0.2 \frac{lb}{min}$ as you increase minutes of excercise. In other words, for each additional minute you will loose approximately 0.2 lb.

(e) $\underset{\substack{\uparrow \\ cal}}{w(2100}, \underset{\substack{\uparrow \\ min}}{20)} = \underset{\substack{\uparrow \\ lb}}{120}$, $\frac{\partial w}{\partial m}(2100, 20) = -0.2 \frac{lb}{min}$

Given that, estimate:

$w(2100, 25)$

$w(2100, 25) \cong 120 \, lb + (-0.2) \frac{lb}{min} \cdot 5 \, min = 119 \, lb$

Approximately as $\frac{\partial w}{\partial m}(2100, 20) = -0.2 \frac{lb}{min}$ is the instantaneous rate at $c = 2100$ and $m = 20$ and it may not stay coutant between $(2100, 20)$ and $(2100, 25)$.

Ex

For Problems (a),(b) refer to Table 9.5 giving the wind-chill factor, C in °F, as a function $f(w,T)$ of the wind speed, w, and the temperature, T. The wind-chill factor is a temperature which tells you how cold it feels, as a result of the combination of wind and temperature.

TABLE 9.5 *Wind-chill factor (°F)* ← T →

↑ w ↓

mph\°F	35	30	25	20	15	10	5	0
5	33	27	21	16	12	7	0	−5
10	22	16	10	3	−3	−9	−15	−22
15	16	9	2	−5	−11	−18	−25	−31
20	12	4	−3	−10	−17	−24	−31	−39
25	8	1	−7	−15	−22	−29	−36	−44

(a) Estimate $f_w(10,25)$. What does your answer mean in practical terms?
(b) Estimate $f_T(5,20)$. What does your answer mean in practical terms?

temperature outside in °F

$$C = f(w, T)$$

wind-chill in °F

wind speed in mph

(a) $f_w(10, 25)$

T stays constant at 25°F, wind speed changes.
The best estimate:

Increasing w: $\dfrac{2\,°F - 10\,°F}{5\text{ mph}} = -\dfrac{8}{5}\,\dfrac{°F}{\text{mph}}$

Decreasing w: $\dfrac{10\,°F - 21\,°F}{5\text{ mph}} = -\dfrac{11}{5}\,\dfrac{°F}{\text{mph}}$

$$f_w(10, 25) \cong -\frac{8}{5}\,\frac{°F}{\text{mph}} = -1.6\,\frac{°F}{\text{mph}}$$

$$\cong -\frac{11}{5}\,\frac{°F}{\text{mph}} = -2.2\,\frac{°F}{\text{mph}}$$

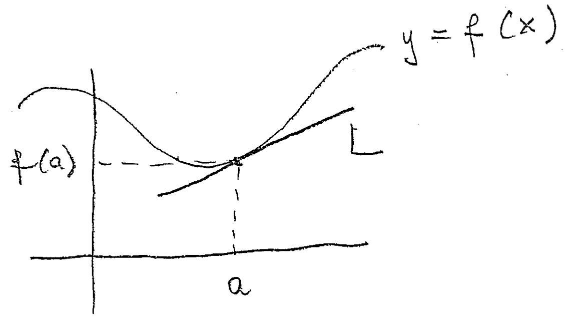
Or average them:

$$f_w(10, 25) \cong \frac{-3.8}{2} = 1.9\,\frac{°F}{\text{mph}}$$

Similarly partial derivatives can be estimated for a given function $z = f(x,y)$ from its contour diagram. Do practice problems for 14.1, 14.2!

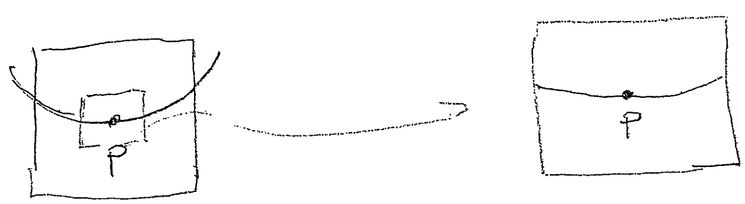
14.3 Differentiability, Tangent Plane, Local Linearization

A function of one variable, $y = f(x)$, is called <u>differentiable</u> at $x = a$ if $f'(a)$ exists which is when the graph $y = f(x)$ has the <u>tangent line</u> at $(a, f(a))$:

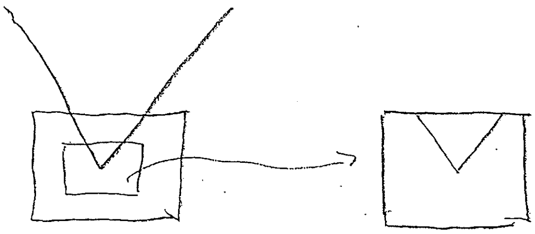


$$f'(a) = m_{tan}^{(a, f(a))}$$

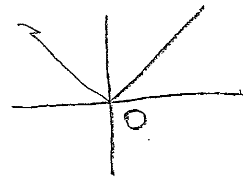
The tangent line exists if the graph of $y = f(x)$ "flattens" to a straight line when we look at smaller and smaller portions of the graph around the point $P = (a, f(a))$:



Not every curve around each point "flattens" to a straight line:



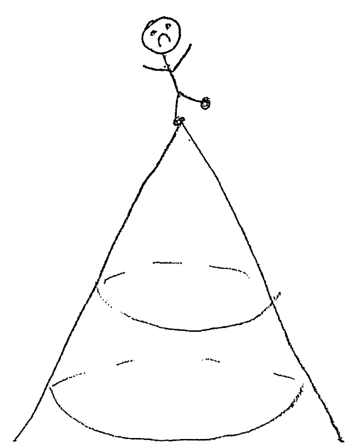
Ex: $f(x) = |x|$ at $a = 0$:



No tangent line.

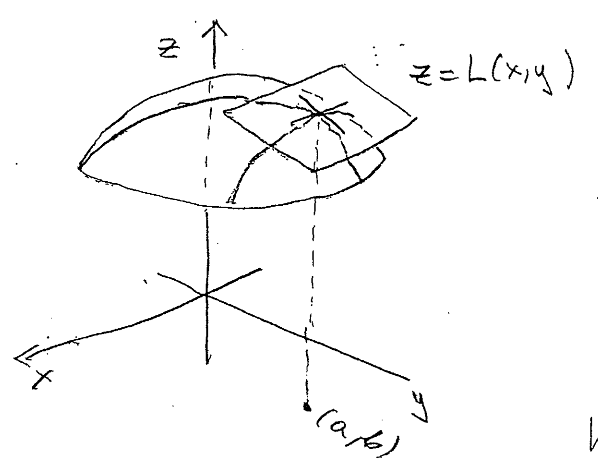
Curves that "flatten" to straight lines as we look at smaller and smaller portions are called "smooth".

How about surfaces in 3D and graphs of functions $z = f(x, y)$? A "nice", "smooth" surface "flattens" to a plane when we look at small portions of it:



That plane it calls the tangent plane.

Let $z = f(x,y)$, $(x,y) = (a,b)$ be given.
$f(x,y)$ is called <u>differentiable</u> at (a,b) if
$z = f(x,y)$ <u>has a tangent plane</u> at $(a, b, f(a,b))$:



Let's denote by $L(x,y)$ the linear function whose graph
$z = L(x,y)$
is the tangent plane.
What is the formula for $L(x,y)$; that is,
what is the equation of the tangent plane?

The tangent lines to the cross-sections $f(x,b)$ and $f(a,y)$ have to be on the tangent plane. Those tangent lines have slopes $f_x(a,b)$ and $f_y(a,b)$. Thus, the slope of $z = L(x,y)$ in the x-direction is $f_x(a,b)$, the slope in the y direction is $f_y(a,b)$. The plane passes through $(a, b, f(a,b))$. Thus the equation of the tanget plane at (a,b) is:

$$z = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b).$$

The linear function whose graph is the tangent plane:

$$L(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b).$$

Clearly:

$$f(x,y) \cong L(x,y) \text{ for } (x,y) \text{ close to } (a,b).$$

$L(x,y)$ is called <u>the local linearization</u> of $f(x,y)$ at (a,b).

<u>Ex</u>: Find the equation of the tangent plane (assuming it exists) to

$$z = f(x,y), \quad f(x,y) = ye^{\frac{x}{y}} \text{ at } (1,1,e).$$

$$z = e + f_x(1,1)(x-1) + f_y(1,1)(y-1)$$

$$f_x = \frac{\partial}{\partial x}\left[ye^{\frac{x}{y}}\right] = \underline{e^{\frac{x}{y}}}, \quad f_x(1,1) = e$$

$$f_y = \frac{\partial}{\partial y}\left[ye^{\frac{x}{y}}\right] = e^{\frac{x}{y}} + y\cdot\left(-\frac{x}{y^2}\right)e^{\frac{x}{y}} = \underline{e^{\frac{x}{y}} - \frac{x}{y}e^{\frac{x}{y}}}$$

$$f_y(1,1) = e - e = 0$$

$$L: \quad z = e + e(x-1) = ex \qquad \boxed{z = ex}$$

Since

$$f(x,y) \cong L(x,y) \quad \text{for } (x,y) \text{ close to } (a,b),$$

we have the following approximation formula:

$$\underline{f(x,y) \cong f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b)}$$

for $(x-a)$, $(y-b)$ "small".

In other words:

$$f(a+\Delta x, b+\Delta y) \cong f(a,b) + f_x(a,b)\Delta x + f_y(a,b)\Delta y,$$

$$\Delta x = x-a, \quad \Delta y = y-b.$$

<u>Ex</u>: An unevenly heated plate has temperature $T(x,y)$ in °C at the point (x,y). If

$$T(2,1) = 135, \quad T_x(2,1) = 16, \quad T_y(2,1) = -15,$$

estimate the temperature at $(2.04, 0.97)$.

$$T(2.04, 0.97) \cong T(2,1) + T_x(2,1)\cdot\Delta x + T_y(2,1)\cdot\Delta y =$$

$\Delta x = 0.04$ $\quad$ $\Delta y = -0.03$

$$\doteq 135 + 16 \cdot 0.04 + (-15)\cdot(-0.03) = \underline{136.09\,^\circ C}$$

The Differential

Let $z = f(x,y)$. We have:

$$f(x,y) \cong f(a,b) + f_x(a,b)\cdot\Delta x + f_y(a,b)\cdot\Delta y$$

Denote

$$\Delta f = f(x,y) - f(a,b).$$

Then:

$$\Delta z = \Delta f \cong f_x(a,b)\cdot\Delta x + f_y(a,b)\cdot\Delta y$$

The infinitesimal version of this formula is called the differential, df or dz, of $f(x,y)$ at (a,b):

$$df = f_x(a,b)\,dx + f_y(a,b)\,dy$$

In general:

$$df = f_x\,dx + f_y\,dy$$

Ex: Compute the differential of

$$z = x\sin(xy).$$

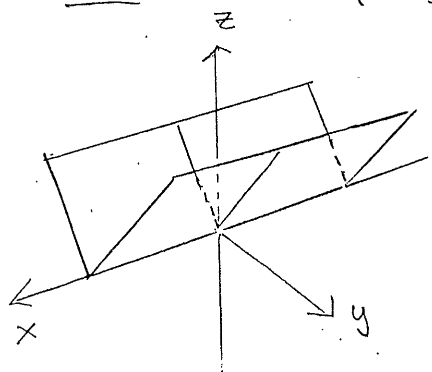
$$\frac{\partial z}{\partial x} = \sin(xy) + xy\cos(xy), \quad \frac{\partial z}{\partial y} = x^2\cos(xy)$$

Thus:

$$dz = (\sin(xy) + xy\cos(xy))\,dx + (x^2\cos(xy))\,dy$$

When is a given function $z = f(x,y)$ differentiable at (a,b)? Is it enough that $f_x(a,b)$, $f_y(a,b)$ exist? Not really. Let's look at a couple of examples.

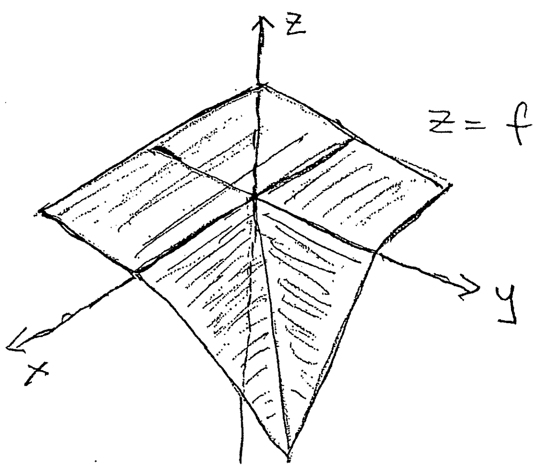
<u>Ex</u> $z = f(x,y)$, $f(x,y) = |y|$.



$f_x(0,0) = ?$ $f_y(0,0) = ?$

Look at cross-sections: $f(x,0)$, $f(0,y)$.

Clearly, $f_x(0,0) = 0$, $f_y(0,0)$ does not exist.
No tangent plane at $(0,0)$ (or any point $(a,0)$).
It can happen, however, that $f_x(a,b)$, $f_y(a,b)$ both exist and there is no tangent plane:



$z = f(x,y)$. What about $f_x(0,0)$, $f_y(0,0)$?
What about differentiability of $f(x,y)$ at $(0,0)$?

Differentiability for $f(x,y)$ is a bit complicated.