MTH 535 F06 Homework 9 – Selected Solutions

1) From Fatou's Lemma we deduce

$$\int_{\mathbb{R}} f \leq \underline{\lim} \int_{\mathbb{R}} f_n . \quad (1)$$

($\int f$ denotes $\int_{\mathbb{R}} f$). The condition $f_n \leq f$ implies via Th 21.1

$$\int_{\mathbb{R}} f_n \leq \int_{\mathbb{R}} f \quad \text{for all } n = 1, 2, \ldots$$

Hence,

$$\overline{\lim} \int_{\mathbb{R}} f_n \leq \int_{\mathbb{R}} f . \ (2)$$

By Th 14.3, $\overline{\lim} \int_{\mathbb{R}} f_n \geq \underline{\lim} \int_{\mathbb{R}} f_n$. Thus, (1), (2) and Th 14.4 imply that $\underline{\lim} \int_{\mathbb{R}} f_n = \overline{\lim} \int_{\mathbb{R}} f_n = \lim \int_{\mathbb{R}} f_n = \int_{\mathbb{R}} f$.

2) Done in class.

3) By definition of f:

$$f = \lim_{k \to +\infty} \sum_{n=1}^{k} s_n .$$

Denote $f_k = \sum_{n=1}^{k} s_n$. Since s_n are nonnegative, $f_{k+1} \geq f_k \geq 0$ for $k = 1, 2, \ldots$ and $f_k \to f$. By the MCT and Th 21.1 (6), we obtain

$$\int_E f = \lim_{k \to +\infty} \int_E f_k = \lim_{k \to +\infty} \int_E \sum_{n=1}^{k} s_n = \lim_{k \to +\infty} \sum_{n=1}^{k} \int_E s_n = \sum_{n=1}^{\infty} \int_E s_n .$$

4) Let $s_n = f \cdot \chi_{E_n}$ for $n = 1, 2, \ldots$ Since $\langle E_n \rangle$ are disjoint on $\bigcup_n E_n = E$ we have:

$$f = \sum_{n=1}^{\infty} s_n \quad \text{on } E.$$

Since f is nonnegative, so are s_n, $n = 1, 2, \ldots$ From the previous problem

$$\int_E f = \sum_{n=1}^{\infty} \int_E s_n = \sum_{n=1}^{\infty} \int_E f \cdot \chi_{E_n} = \sum_{n=1}^{\infty} \int_{E_n} f .$$

5) Since $f_n \geq f_{n+1} \geq 0$ on E, we have $0 \leq f_n \leq f_1$ on E for $n=1,2,\ldots$. Thus, $|f_n| \leq f_1$ on E for $n=1,2,\ldots$, and f_1 is integrable on E. We can apply the LDCT to the sequence $\langle f_n \rangle$ and obtain $\int_E f_n \to \int_E f$.

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6) As $f_n \to f$ a.e. in $[0,1]$, $|f_n - f| \longrightarrow 0$ a.e. in $[0,1]$.
Also for $n=1,2,\ldots$:

$$|f_n - f| \leq |f_n| + |f| \leq g + |f| \quad \text{a.e. in } [0,1].$$

But f is integrable in $[0,1]$ and so is $|f|$.
Thus, $g + |f|$ is integrable in $[0,1]$ as a sum of two integrable functions. We can apply the LDCT (more precisely, Prop. 23.3) to the sequence $|f_n - f|$ and deduce

$$\int_{[0,1]} |f_n - f| \to \int_{[0,1]} 0 = 0.$$
