<u>Th 17.1 (Egoroff)</u>: Let $f, f_n : D \to \mathbb{R}$, $n = 1, 2, \dots$, be measurable, $m(D) < +\infty$. Assume that $f_n \to f$ a.e. on D. Then for every $\varepsilon > 0$ there exists $E_\varepsilon \subseteq D$ such that $E_\varepsilon \in \mathcal{M}$, $m(E_\varepsilon) < \varepsilon$ and $f_n \rightrightarrows f$ on $D \sim E_\varepsilon$.

▲

Before we prove the theorem recall the following two propositions:

(P1) If f is measurable, then $|f|$ is measurable.

Indeed, $|f| = f^+ + f^-$. f^+, f^- are measurable. (By # 5, H7.)

(P2) If $A_1, A_2, \dots, A_n, \dots \in \mathcal{M}$, $A_{n+1} \supseteq A_n$ for $n = 1, 2, \dots$, then

$$m\left(\bigcup_{n=1}^{\infty} A_n\right) = \lim_{n \to +\infty} m(A_n).$$

(By # 2, H6.)

<u>Proof of Th 17.1</u>: Let $\varepsilon > 0$ be fixed. We want to show that there exists $E_\varepsilon \subseteq D$ such that $m(E_\varepsilon) < \varepsilon$ and

$$\forall_{\eta > 0} \; \exists_{N_\eta \in \mathbb{N}} \; \forall_{n \geq N_\eta} \; \forall_{x \in D \sim E_\varepsilon} \quad |f_n(x) - f(x)| < \eta. \quad (1)$$

In other words, $f_n \rightrightarrows f$ on $D \sim E_\varepsilon$.

Let $D_0 \subseteq D$ be:

$$D_0 = \{x \in D : f_n(x) \not\to f(x)\}.$$

Then $D_0 \in \mathcal{M}$, $m(D_0) = 0$. For each pair $n \in \mathbb{N}$ and $i \in \mathbb{N}$ define the following set:

$$E_{i,n} = \{x \in D \sim D_0 : |f_m(x) - f(x)| < \tfrac{1}{2^i} \text{ for } m \geq n\}. \quad (2)$$

As $|f_m - f|$ is measurable, $E_{i,n} \in \mathcal{M}$ for $i, n \in \mathbb{N}$.

From (2), $E_{i,n+1} \supseteq E_{i,n}$ for all $i, n \in \mathbb{N}$. Also for every $i \in \mathbb{N}$:

$$\bigcup_{n=1}^{\infty} E_{i,n} = D \sim D_0 \quad (3)$$

To prove (3), fix i and take $x \in D \sim D_0$. Then $f_n(x) \to f(x)$, thus, $|f_n(x) - f(x)| < \frac{1}{2^i}$ for $n \geq N$ for some N (which depends on x and i). Hence, $x \in E_{i,N}$ and the inclusion $D \sim D_0 \subseteq \bigcup_{n=1}^{\infty} E_{i,n}$ is proved. The opposite inclusion follows from (2). Thus, (3) is proved.

By (P2), for every fixed $i = 1, 2, \dots$, we have:

$$\lim_{n \to +\infty} m(E_{i,n}) = m(D \sim D_0) = m(D) - m(D_0) = m(D). \quad (4)$$

(The second equality follows from Prop 12.3 as $m(D) < +\infty$.)

From (4), for every fixed i, there exists $N_i \in \mathbb{N}$ such that

$$m(E_{i,N_i}) > m(D) - \frac{\varepsilon}{2^i}, \quad m(D \sim E_{i,N_i}) < \frac{\varepsilon}{2^i}. \quad (5)$$

Define

$$E_\varepsilon = D \sim \bigcap_{i=1}^{\infty} E_{i,N_i} = \bigcup_{i=1}^{\infty} (D \sim E_{i,N_i}) . \quad (6)$$

By the latter equality and (5), we have

$$m(E_\varepsilon) = m\left(\bigcup_{i=1}^{\infty} (D \sim E_{i,N_i})\right) \leq \sum_{i=1}^{\infty} \frac{\varepsilon}{2^i} = \varepsilon .$$

It remains to show that $f_n \rightrightarrows f$ on $D \sim E_\varepsilon$. Observe that by (6):

$$D \sim E_\varepsilon = \bigcap_{i=1}^{\infty} E_{i,N_i}$$

Therefore, for every $x \in D \sim E_\varepsilon$ we have

$$x \in E_{i,N_i} \quad \text{for } i = 1, 2, \ldots .$$

From (2), we obtain that for all $x \in D \sim E_\varepsilon$ and all $i = 1, 2, \ldots$:

$$|f_m(x) - f(x)| < \frac{1}{2^i} \quad \text{for } m \geq N_i . \quad (7)$$

We are ready to show (1). Let $\eta > 0$ be given. Let $i_0 \in \mathbb{N}$ be such that $\frac{1}{2^{i_0}} < \eta$. (7) implies that

$$|f_n(x) - f(x)| < \frac{1}{2^{i_0}} < \eta \quad \text{for all } n \geq N_{i_0}, x \in D \sim E_\varepsilon .$$

Hence, (1) holds with $N_\eta = N_{i_0}$. The proof is complete. ∎