Extremal Problems for Roman Domination

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Abstract

A *Roman dominating function* of a graph G is a labeling $f\colon V(G) \to \{0,1,2\}$ such that every vertex with label 0 has a neighbor with label 2. The *Roman domination number* $\gamma_R(G)$ of G is the minimum of $\sum_{v \in V(G)} f(v)$ over such functions. Let G be a connected n-vertex graph. We prove that $\gamma_R(G) \le 4n/5$, and we characterize the graphs achieving equality. We obtain sharp upper and lower bounds for $\gamma_R(G) + \gamma_R(\overline{G})$ and $\gamma_R(G)\gamma_R(\overline{G})$, improving known results for domination number. We prove that $\gamma_R(G) \le 8n/11$ when $\delta(G) \ge 2$ and $n \ge 9$, and this is sharp. We prove that $\gamma_R(G) \le \lceil 2n/3 \rceil$ for a special class of 2-connected graphs.

1   Introduction

According to [**?**], Constantine the Great (Emperor of Rome) issued a decree in the 4th century A.D. for the defense of his cities. He decreed that any city without a legion stationed to secure it must neighbor another city having two stationed legions. If the first were attacked, then the second could deploy a legion to protect it without becoming vulnerable itself.

The objective, of course, is to minimize the total number of legions needed. The problem generalizes to arbitrary graphs. A *Roman dominating function (RDF)* on a graph G is a vertex labeling $f\colon V(G) \to \{0,1,2\}$ such that every vertex with label 0 has a neighbor with label 2. For an RDF f, let $V_i(f) = \{v \in V(G)\colon f(v) = i\}$. In the context of a fixed RDF, we suppress the argument and simply write V_0, V_1, and V_2. Since this partition determines f, we can equivalently write $f = (V_0, V_1, V_2)$. The *weight* $w(f)$ of an RDF f is $\sum_{v \in V(G)} f(v)$, which equals $|V_1| + 2|V_2|$. The *Roman domination number* $\gamma_R(G)$ is the minimum weight of an RDF of G. Thus, $\gamma_R(G)$ is the minimum number of legions needed to protect cities whose adjacency graph is G.

Roman domination also models other facility location problems. Instead of interpreting $f(v)$ as the number of units placed at v, we can view it as a cost function. Units with cost 2 may be able to serve neighboring locations, while units with cost 1 can serve only their own location. For example, in a communication network, wireless hubs are more expensive but can serve neighboring locations, while wired hubs are low-range but are cheaper.

Cockayne, Dreyer, Hedetniemi, and Hedetniemi [?] began the study of Roman domination, suggested in a *Scientific American* article by Stewart [?] and even earlier by ReVelle [?]. Since $V_1 \cup V_2$ is a dominating set when f is an RDF, and since placing weight 2 at the vertices of a dominating set yields an RDF, [?] observed that

$$\gamma(G) \leq \gamma_R(G) \leq 2\gamma(G), \tag{1}$$

where $\gamma(G)$ is the domination number of G. In a sense, $2\gamma(G) - \gamma_R(G)$ measures "inefficiency" of domination, since when $\gamma_R(G) = (2 - \beta)\gamma(G)$, at least the fraction β of the vertices in a minimum dominating set serve only to dominate themselves.

Cockayne, Dreyer, Hedetniemi, and Hedetniemi [?] studied basic properties of Roman dominating functions and calculated γ_R for specific graphs. They characterized the graphs G such that $\gamma_R(G) \leq \gamma(G) + k$ when $k \leq 2$; this was extended to larger k in [?]. They also characterized graphs G such that $\gamma_R(G) = 2\gamma(G)$ in terms of 2-packings, calling such graphs *Roman*. Henning [?] characterized Roman trees, while Song and Wang [?] characterized the trees T with $\gamma_R(T) = \gamma(T) + 3$. The computational complexity of $\gamma_R(G)$ was studied in [?]. Linear-time algorithms for computing $\gamma_R(G)$ are known on interval graphs [?, ?], cographs [?], and strongly chordal graphs [?]. A polynomial-time algorithm is known on AT-free graphs [?]. Other related domination models were studied in [?, ?, ?, ?, ?].

In this paper, we study extremal problems for $\gamma_R(G)$ on various classes of n-vertex graphs. In Section ??, we prove that $\gamma_R(G) \leq 4n/5$ when G is connected and $n \geq 3$, and we determine when equality holds. In Section ??, we obtain sharp upper and lower bounds for $\gamma_R(G) + \gamma_R(\overline{G})$ and $\gamma_R(G)\gamma_R(\overline{G})$, where $\overline{G}$ denotes the complement of G. We use these ideas to determine the n-vertex graphs G with largest value of $\gamma(G)\gamma(\overline{G})$, shown to equal n in [?].

Let $\delta(G)$ denote the minimum vertex degree in G. When $\delta(G) \geq k$, inequality (??) and the well-known upper bound on $\gamma(G)$ from [?, ?] yield $\gamma_R(G) \leq 2\frac{1+\ln(k+1)}{k+1}n$. This was improved slightly in [?]; we use their improvement in Section ??. For small k, the optimal coefficient is of interest. In Section ??, we prove that if G is a connected n-vertex graph with $\delta(G) \geq 2$ and $n \geq 9$, then $\gamma_R(G) \leq 8n/11$. The bound is sharp, and we determine when equality holds. In Section ??, we prove that $\gamma_R(G) \leq \lceil 2n/3 \rceil$ for a special class of 2-connected graphs.

In an earlier version of this paper, we conjectured that this upper bound holds for all 2-connected graphs. Subsequently, Chang and Liu [?] found 2-connected n-vertex graphs such that $\gamma_R(G) = 23n/34$ for infinitely many n; note that $\frac{23}{34} = \frac{2}{3} + \frac{1}{102}$. They also proved that $\gamma_R(G) \leq \max\{\lceil 2n/3 \rceil, 23n/34\}$ when G is 2-connected, making use of our remark in

Lemma **??**(a). For minimum degree 3, they proved that $\gamma_R(G) \leq 2n/3$ and that this is sharp for infinitely many 3-connected graphs.

Our graphs have no loops or multiple edges; we use $V(G)$ and $E(G)$ for the vertex set and edge set of a graph G. The degree of a vertex v in G is $d_G(v)$ or simply $d(v)$. The minimum and maximum vertex degrees are $\delta(G)$ and $\Delta(G)$. For a set $S \subseteq V(G)$, the *(open) neighborhood* of S is $\{v \in V(G) - S \colon v \text{ has a neighbor in } S\}$, denoted $N(S)$. The *closed neighborhood* of S is $N(S) \cup S$, denoted $N[S]$. When $S = \{v\}$, we simply write $N(v)$ and $N[v]$. The *diameter* of G is the maximum distance between vertices of G, denoted $\operatorname{diam} G$. In a tree, a *penultimate vertex* is any neighbor of a leaf. We write P_n, C_n, and K_n for the path, cycle, and complete graph with n vertices, respectively. We write mG for the graph consisting of m disjoint copies of G.

2   Connected Graphs

For n-vertex graphs, always $\gamma_R(G) \leq n$, with equality when $G = \overline{K}_n$. In this section we prove that $\gamma_R(G) \leq 4n/5$ when G is a connected n-vertex graph and characterize when equality holds. Since $\gamma(G)$ may be as high as $n/2$, (**??**) only gives $\gamma_R(G) \leq n$, so proving the bound of $4n/5$ needs additional work. Since deleting an edge cannot decrease γ_R, it suffices to prove the bound for trees.

Theorem 2.1 *If T is an n-vertex tree, with $n \geq 3$, then $\gamma_R(T) \leq 4n/5$.*

Proof.   We use induction on n. The base step handles trees with few vertices or small diameter. If $\operatorname{diam} T = 2$, then T has a dominating vertex, and $\gamma_R(T) \leq 2 < 4n/5$. If $\operatorname{diam} T = 3$, then T has a dominating set of size 2, which yields $\gamma_R(T) \leq 4$. This is sufficiently small for trees with at least six vertices. For $n \in \{4, 5\}$ and $\operatorname{diam} T = 3$, a penultimate vertex has degree 2; putting weight 2 on the other penultimate vertex and weight 1 on the undominated leaf yields $\gamma_R(T) \leq 3$, which is small enough.

Hence we may assume that $\operatorname{diam} T \geq 4$. For a subtree T' with n' vertices, where $n' \geq 3$, the induction hypothesis yields an RDF f' of T' with weight at most $\frac{4}{5}n'$. We find a subtree T' such that adding a bit more weight to f' will yield a small enough RDF f for T.

Let P be a longest path in T chosen to maximize the degree of its next-to-last vertex v, and let u be the non-leaf neighbor of v.

Case 1: $d_T(v) > 2$. Obtain T' by deleting v and its leaf neighbors. Since $\operatorname{diam} T \geq 4$, we have $n' \geq 3$. Define f on $V(T)$ by letting $f(x) = f'(x)$ except for $f(v) = 2$ and $f(x) = 0$ for each leaf x adjacent to v. Note that f is an RDF for T and that $w(f) = w(f') + 2 \leq \frac{4}{5}(n-3) + 2 < \frac{4}{5}n$.

Case 2: $d_T(v) = d_T(u) = 2$. Obtain T' by deleting u and v and the leaf neighbor z of v. If $n' = 2$, then T is P_5 and has an RDF of weight 4. Otherwise, the induction hypothesis

applies. Define f on $V(T)$ by letting $f(x) = f'(x)$ except for $f(v) = 2$ and $f(u) = f(z) = 0$. Again f is an RDF, and the computation $w(f) < \frac{4}{5}n$ is the same as in Case 1.

Case 3: $d_T(u) > 2$ *and every penultimate neighbor of u has degree 2.* If every neighbor of u is penultimate or a leaf, then $\operatorname{diam} T = 4$ and T is obtained from a star with center u by subdividing k edges, where $k \geq 2$. Put weight 2 on u and weight 1 on the non-neighbors of u. Now $w(f) = k + 2$ and $n \geq 2k + 1 \geq 5$, so $w(f) \leq (n+3)/2 \leq \frac{4}{5}n$.

Otherwise, some neighbor t of u is neither penultimate nor a leaf. Obtain T' from T by deleting the vertices of the component of $T - tu$ containing u. Now $n' \geq 3$ and the induction hypothesis applies. Define f on $V(T)$ by $f(x) = f'(x)$ except for $f(u) = 2$, $f(x) = 1$ for each non-neighbor x of u outside T', and $f(x) = 0$ for $x \in N(u) - \{t\}$. Again f is an RDF. We have $w(f) = w(f') + k + 2$, where k is the number of leaves of T at distance 2 from u.

If $k = 1$, then $d_T(u) > 2$ forces u to have a leaf neighbor, and $w(f) \leq \frac{4}{5}(n-4) + 3 < \frac{4}{5}n$. Otherwise $k \geq 2$, and $w(f) \leq \frac{4}{5}(n - 2k - 1) + (k + 2) = \frac{1}{5}(4n - 3k + 6) \leq \frac{4}{5}n$. □

As shown in [**?**], $\gamma_R(P_n) \leq (2n + 2)/3$. The path is not the worst-case n-vertex tree; equality in Theorem **??** is achievable. Let L_k consist of the disjoint union of k copies of P_5 plus a path through the central vertices of these copies, as illustrated in Figure **??**.
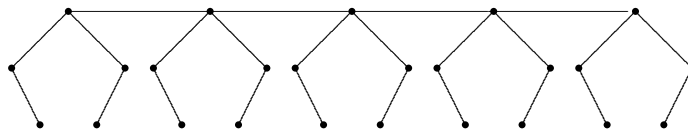

Figure 1: The tree L_5.

If u is a vertex of degree 2 having a leaf neighbor v, then an RDF must put total weight at least 2 on $\{u, v\}$ unless the other neighbor of u has weight 2. Thus when two such vertices u and u' have a common neighbor w, an RDF must give total weight at least 4 to $\{v, u, w, u', v'\}$. In L_k, there are k disjoint 5-vertex sets of this form, so $\gamma_R(L_k) \geq 4k = 4n/5$. Such copies of P_5 can be assembled in many ways, and this allows us to characterize the trees achieving equality in Theorem **??**.

Theorem 2.2 *If T is an n-vertex tree, then $\gamma_R(T) = 4n/5$ if and only if $V(T)$ can be partitioned into sets inducing P_5 such that the subgraph induced by the central vertices of these paths is connected.*

Proof. We have observed that if an induced subgraph H of G is isomorphic to P_5, and its noncentral vertices have no neighbors outside H in G, then every RDF of G puts weight at least 4 on $V(H)$. Thus in any tree with such a vertex partition, weight at least 4 is needed on every set in the partition.

To show that equality requires this structure, we examine the proof of Theorem **??** more closely. The proof is by induction on n. In the base cases and Cases 1 and 2, we produce an

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RDF with weight less than $4n/5$. In Case 3 with diameter 4, equality requires $n = 2k + 1$ and $k = 2$, and the only such tree is P_5 itself.

Define u, T', n', t, k as in the inductive part of Case 3. The bound holds with equality only if $k = 2$ and $n' = n - (2k + 1)$. Thus u has no leaf neighbors, and $T - V(T')$ is a 5-vertex path Q with center u. Equality also requires $\gamma_R(T') = 4n'/5$, so by the induction hypothesis T' has the specified form. In particular, t lies in a copy P' of P_5 in a covering of $V(T')$ by 5-sets inducing paths. Let t' be the center of P'.

If $t \neq t'$, then we build a cheaper RDF for T. Put weight 2 on u and weight 1 on the leaves of Q. Put weight 1 on the neighbor of t in $T' - t'$, and put weight 2 on the penultimate vertex of P' farthest from t. We have now guarded $P' \cup Q$ using total weight 7, and hence $\gamma_R(T) < \frac{4}{5}n$. Hence equality requires $t = t'$ and the specified structure for T. $\qquad\square$

It is easy to extend this characterization to all connected graphs.

Theorem 2.3 *If G is a connected n-vertex graph, then $\gamma_R(G) \leq 4n/5$, with equality if and only if G is C_5 or is obtained from $\frac{n}{5}P_5$ by adding a connected subgraph on the set of centers of the components of $\frac{n}{5}P_5$.*

Proof. If G has the specified form, then as remarked earlier every RDF puts weight at least 4 on the vertex set of each copy of P_5.

Now suppose that $\gamma_R(G) = \frac{4}{5}n$. Since adding edges cannot increase γ_R, every spanning tree of G has the form specified in Theorem **??**. Given a spanning tree T, let $S_1, \ldots, S_k$ be the 5-sets in the special partition of $V(T)$. The assignment of weight 4 that guards S_i can be chosen independently of any other S_j. If any edge of G joins vertices of S_i and S_j that are not the centers of the paths they induce, then an RDF with weight less than $\frac{4}{5}n$ can be built as in the proof of Theorem **??**. $\qquad\square$

3   Nordhaus-Gaddum Inequalities

For a graph parameter ρ, bounds on $\rho(G) + \rho(\overline{G})$ and $\rho(G)\rho(\overline{G})$ in terms of the number of vertices are called results of "Nordhaus–Gaddum" type, honoring the paper of Nordhaus and Gaddum [**?**] obtaining such bounds when ρ is the chromatic number.

For an n-vertex graph G with $n \geq 2$, it is known (see [**?**, p. 237]) that

$$3 \leq \gamma(G) + \gamma(\overline{G}) \leq n + 1 \tag{2}$$

$$2 \leq \gamma(G)\gamma(\overline{G}) \leq n. \tag{3}$$

In this section we obtain the analogous sharp results for γ_R.

Proposition 3.1 *If G is an n-vertex graph, then $\gamma_R(G) \leq n - \Delta(G) + 1$.*

Proof. When v is a vertex of maximum degree, the RDF $(N(v), V(G) - N[v], \{v\})$ has weight $n - \Delta(G) + 1$. $\square$

Theorem 3.2 *If G is an n-vertex graph, with $n \geq 3$, then*

$$5 \leq \gamma_R(G) + \gamma_R(\overline{G}) \leq n + 3.$$

Furthermore, equality holds in the upper bound only when G or $\overline{G}$ is C_5 or $\frac{n}{2}K_2$.

Proof. When G has at least three vertices, $\gamma_R(G) \geq 2$, with equality only when G has a dominating vertex. Since a graph and its complement cannot both have dominating vertices, $\gamma_R(G) + \gamma_R(\overline{G}) \geq 5$. Equality holds if and only if G or $\overline{G}$ has a vertex of degree $n - 1$ and its complement has a vertex of degree $n - 2$.

For the upper bound, Proposition **??** yields

$$\gamma_R(G) + \gamma_R(\overline{G}) \leq (n - \Delta(G) + 1) + (n - \Delta(\overline{G}) + 1)$$
$$= n - \Delta(G) + \delta(G) + 3 \leq n + 3.$$

If $\gamma_R(G) + \gamma_R(\overline{G}) = n + 3$, then equality holds throughout the calculation, and $\delta(G) = \Delta(G)$. Hence G is k-regular for some k. We may assume that $k \leq (n - 1)/2$, since the argument is symmetric in G and $\overline{G}$. Since equality holds, $\gamma_R(G) = n - k + 1$ and $\gamma_R(\overline{G}) = k + 2$.

Let $v \in V(G)$. If some vertex u outside $N[v]$ has at least two neighbors outside $N[v]$, then the RDF $(N(u) \cup N(v), V(G) - N[u] - N[v], \{u, v\})$ has weight at most $n - k$, a contradiction. Hence every vertex not in $N[v]$ has at least $k - 1$ neighbors in $N(v)$. Similarly, each vertex in $N(v)$ has at most two neighbors outside $N[v]$.

Counting the edges joining $N(v)$ and $V(G) - N[v]$ from both sides yields $(k-1)(n-k-1) \leq 2k$, simplifying to $n \leq k + 3 + \frac{2}{k-1}$ for $k > 1$. Since $n \geq 2k + 1$, we have $k \leq 2 + \frac{2}{k-1}$, which requires $k \leq 3$. If $k = 3$, then $n = 7$, but there is no 3-regular 7-vertex graph.

For $k = 2$, we have $n \leq k + 3 + \frac{2}{k-1} = 7$ and $n \geq 2k + 1 = 5$. For each 2-regular graph G with $n \in \{6, 7\}$, we have $\gamma_R(G) = n - 2$, so $\gamma_R(G) = n - k + 1$ leaves only $G = C_5$.

For $k = 1$, the only example is $\frac{n}{2}K_2$, where equality holds. For $k = 0$, the only example is $G = \overline{K}_n$, where $\gamma_R(G) + \gamma_R(\overline{G}) = n + 2$, and equality does not hold. $\square$

For the product bound, (**??**) and (**??**) yield $\gamma_R(G)\gamma_R(\overline{G}) \leq 4n$. The optimal bound is smaller for sufficiently large n. We will prove in Theorem **??** that $\gamma_R(G)\gamma_R(\overline{G}) \leq 16n/5$ when $n \geq 160$. Sharpness is shown by $G = kC_5$, since $\gamma_R(kC_5) = 4k$ and $\gamma_R(\overline{kC_5}) = 4$ and $|V(kC_5)| = 5k$. In fact, equality holds only when G or $\overline{G}$ is kC_5 (when n is large).

The most difficult case in the proof of Theorem **??** is when $\operatorname{diam} G = \operatorname{diam} \overline{G} = 2$. We handle this case separately in the next lemma, using a result from Cockayne, Dreyer, Hedetniemi, and Hedetniemi [**?**]. For an n-vertex graph G, they proved that

$$\gamma_R(G) \leq \frac{2 + 2\ln((1 + \delta(G))/2)}{1 + \delta(G)}n. \tag{4}$$

Since $\gamma_R(G) \le 2\gamma(G)$, this bound slightly refines the well-known bound $\gamma(G) \le \frac{1+\ln(1+\delta(G))}{1+\delta(G)} n$ due to Arnautov [**?**] and Payan [**?**].

Lemma 3.3 *If G is an n-vertex graph with $n \ge 160$, and $\operatorname{diam} G = \operatorname{diam} \overline{G} = 2$, then $\gamma_R(G)\gamma_R(\overline{G}) < 16n/5$.*

Proof. Let G be such a graph, and let v be a vertex of minimum degree in G. If $d(v) \le 2$, then the diameter constraint implies that $(V(G) - N(v), \varnothing, N(v))$ is an RDF of G and that $(V(G) - N[v], N(v), v)$ is an RDF of $\overline{G}$, so $\gamma_R(G)\gamma_R(\overline{G}) \le 16$. Hence we may assume that $d_G(v) \ge 3$, and similarly $\delta(\overline{G}) \ge 3$.

Let $R = V(G) - N_G[v]$. We select disjoint subsets of $N_G(v)$ dominating R as follows. Initialize $B_0 = N_G(v)$; note that B_0 dominates R, since $\operatorname{diam} G = 2$. If B_i dominates R, then let A_{i+1} be a minimal subset of B_i dominating R, let $B_{i+1} = B_i - A_i$, and increment i. If B_i does not dominate R, then stop, setting $q = i$ and $A^* = B_q$. Thus $A_1, \ldots, A_q$ partition $N_G(v) - A^*$, with each A_i being a minimal set that dominates R.

Since A_i is a minimal set that dominates R, there is a vertex $r_i \in R$ having only one neighbor in A_i; let a_i be this neighbor. Since A^* does not dominate R, there exists $w \in R$ such that $A^* \subseteq N_{\overline{G}}(w)$. Let $S = \{r_1, \ldots, r_q\} \cup \{v, w\}$ and $T = \{a_1, \ldots, a_q\}$. Now $(V(G) - (S \cup T), T, S)$ is an RDF for $\overline{G}$, since v dominates R, w dominates A^*, and r_i dominates $A_i - \{a_i\}$. Thus $\gamma_R(\overline{G}) \le 3q + 4$, which reduces to $3q + 2$ if $A^* = \varnothing$.

Let $U = A_j \cup \{v\}$, where $|A_j| = \min_i |A_i|$. Note that U is a dominating set of G. If $|U| = 2$, then $\gamma_R(G) \le 4$. Since $\overline{G}$ is connected and $\delta(\overline{G}) \ge 3$, Theorem **??** yields $\gamma_R(\overline{G}) < 4n/5$. Hence we may assume that $|U| > 2$, which requires $q \le \delta(G)/2$.

If $q = 1$, then $\gamma_R(\overline{G}) \le 7$ and $\gamma_R(G) \le 2|U| \le 2(\delta(G)+1)$, so $\gamma_R(G)\gamma_R(\overline{G}) \le 14(\delta(G)+1)$. Hence we may assume in this case that $\delta(G) \ge 8n/35 - 1$, but now (**??**) yields $\gamma_R(G) \le \frac{1+\ln(4n/35)}{4/35}$. Since $7 \cdot \frac{1+\ln(4n/35)}{4/35} < \frac{16n}{5}$ when $n \ge 54$, we have $\gamma_R(G)\gamma_R(\overline{G}) < 16n/5$.

Hence we may assume that $2 \le q \le \delta(G)/2$. Using the RDF $(V(D) - U, \varnothing, U)$ and maximizing over $2 \le q \le \delta(G)/2$ (which requires $\delta(G) \ge 4$) yields

$$\gamma_R(G)\gamma_R(\overline{G}) \le \left(\frac{2\delta(G)}{q} + 2 \right)(3q + 4) = (6\delta(G) + 8) + (6q + \frac{8\delta(G)}{q}) \le 10\delta(G) + 20. \quad (5)$$

Since $10\delta(G) + 20 < 16n/5$ when $\delta(G) + 2 < 8n/25$, we may assume that $\delta(G) \ge 8n/25 - 2$, and similarly for $\delta(\overline{G})$. By (**??**), $\max\{\gamma_R(G), \gamma_R(\overline{G})\} \le \frac{2+2\ln(4n/25)}{8n/25 - 1} n$. With $n \ge 160$, this bound is less than $16n/95$.

If $q \le 5$, then $\gamma_R(\overline{G}) \le 19$. If $q \ge \delta(G)/8$, then $\gamma_R(G) \le 18$. In these cases we obtain $\gamma_R(G)\gamma_R(\overline{G}) < \frac{16n}{95} \cdot 19 = 16n/5$.

Hence we may assume that $6 \le q \le \delta(G)/8$. Now $(2\delta(G)/q + 2)(3q + 4) \le 22\delta(G)/3 + 44$, since $\delta(G) \ge 48$. This bound is less than $16n/5$ when $\delta(G) < 24n/55 - 6$, so we may assume that $\delta(G)$ and $\delta(\overline{G})$ are at least $24n/55 - 6$. Now (**??**) yields

$$\gamma_R(G)\gamma_R(\overline{G}) \le \left(\frac{(2 + 2\ln(12n/55))n}{24n/55 - 5} \right)^2 .$$

The upper bound is less than $16n/5$ when $n \geq 141$. $\qquad\square$

This method of proof actually yields $\gamma_R(G)\gamma_R(\overline{G}) = O((n \ln n)^{2/3})$ when $\operatorname{diam} G = \operatorname{diam} \overline{G} = 2$.

Theorem 3.4 *If G is an n-vertex graph and $n \geq 160$, then*

$$\gamma_R(G)\gamma_R(\overline{G}) \leq \frac{16n}{5},$$

with equality only when G or $\overline{G}$ is $\frac{n}{5}C_5$.

Proof. If G has an isolated vertex or edge, then $\gamma_R(\overline{G}) \leq 3$, which yields $\gamma_R(G)\gamma_R(\overline{G}) \leq 3n < 16n/5$. Thus we may assume that each component of G has at least three vertices. Applying Theorem **??** to each component now yields $\gamma_R(G) \leq 4n/5$.

If $\operatorname{diam} G \geq 3$, then G has vertices u and v with no common neighbor. Hence $\{u,v\}$ is a dominating set in $\overline{G}$, and $\gamma_R(\overline{G}) \leq 4$. Thus $\gamma_R(G)\gamma_R(\overline{G}) \leq (4n/5)4$ when $\operatorname{diam} G \geq 3$, and similarly when $\operatorname{diam} \overline{G} \geq 3$. Lemma **??** produces the desired bound in the remaining case.

Since Lemma **??** establishes strict inequality, the only way to achieve equality in this bound is if $\gamma_R(G) = 4n/5$ and $\gamma_R(\overline{G}) = 4$ (or vice versa). If $\gamma_R(\overline{G}) = 4$, then $\delta(G) \geq 2$, so Theorem **??** implies that every component of G is a 5-cycle. $\qquad\square$

A similar analysis gives the analogous result for domination number.

Theorem 3.5 *If G is an n-vertex graph, with $n \geq 184$, then equality holds in the bound $\gamma(G)\gamma(\overline{G}) \leq n$ of (**??**) if and only if $\gamma(G)$ or $\gamma(\overline{G})$ equals n or $n/2$.*

Proof. If G or $\overline{G}$ is K_n, then equality holds.

If $\delta(G) = 1$, then $\gamma(\overline{G}) = 2$, and equality holds if and only if $\gamma(G) = n/2$. It is known (see [**?**]) that an n-vertex graph G without isolated vertices has domination number $n/2$ if and only if $G = C_4$ or G is obtained from some graph with $n/2$ vertices by adding a pendant edge to each vertex. Thus if $n > 4$ and $\gamma(G) = n/2$, then $\gamma(G)\gamma(\overline{G}) = n$.

For $\delta(G) \geq 2$, McQuaig and Shepherd [**?**] proved that $\gamma(G) \leq 2n/5$. If also $\operatorname{diam} \overline{G} \geq 3$, then $\gamma(G)\gamma(\overline{G}) \leq 4n/5 < n$. Hence we may assume that both G and $\overline{G}$ have diameter 2.

When $\operatorname{diam} G = \operatorname{diam} \overline{G} = 2$, essentially the same argument (with obvious changes) as in the proof of Lemma **??** shows that $\gamma(G)\gamma(\overline{G}) < n$ for $n \geq 184$. We omit the details. $\qquad\square$

4   Minimum Degree 2

In this section, we consider how large γ_R can be for connected n-vertex graphs with minimum degree at least 2. In the n-vertex graph G illustrated in Figure **??**, an RDF must give weight 4 to an induced 5-cycle unless one of its vertices has an outside neighbor with weight 2.

When there is one such vertex, deleting it from the 5-cycle leaves a 4-vertex path that still needs weight 3 on it to be guarded. Hence each subgraph formed from two 5-cycles and a common neighbor must receive weight at least 8, and we obtain $\gamma_R(G) = 8n/11$.
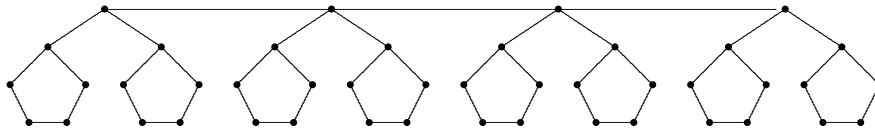

Figure 2: n-vertex graph G with $\gamma_R(G) = 8n/11$.

Lemma 4.1 *Let G be a graph with $\delta(G) \geq 2$. If G contains any configuration listed below, then there exists G' such that $\delta(G') \geq 2$, $|V(G')| = |V(G)| - 3$, and $\gamma_R(G) \leq \gamma_R(G') + 2$.*
a) An induced 5-vertex path C whose internal vertices have degree 2 in G.
b) An induced 4-cycle C whose vertices alternate between degree 2 and larger degree in G.
c) An induced 6-cycle C with exactly two vertices having degree at least 3 in G.

Proof. In each case, we define a graph G' with three fewer vertices than G such that $\delta(G') \geq 2$, let f' be an RDF of G', and produce an RDF f of G with $w(f) \leq w(f') + 2$. Let x and y be the two vertices of C having at least one neighbor outside $V(C)$.

(a) Let the vertices of C be x, u, v, w, y in order. Form G' from G by deleting $\{u, v, w\}$ and adding the edge xy. Since x and y are nonadjacent in G, every vertex of G' has the same degree in G' as in G. Let $f(v) = 2$ and $f(u) = f(w) = 0$, with $f(z) = f'(z)$ for $z \in V(G')$. This suffices unless $\{f'(x), f'(y)\} = \{2, 0\}$ and the edge xy is needed for f' to be an RDF. By symmetry, we may assume $f'(y) = 0$; in this case, let $f(w) = 2$ instead of $f(v) = 2$.

(b) Contract C into a single vertex v to form G', so $d_{G'}(v) \geq 2$ and $\delta(G') \geq 2$. Let u and w be the vertices of C with degree 2 in G. Let $f(z) = f'(z)$ for $z \in V(G) - V(C)$. If $f'(v) \in \{1, 2\}$, then let $f(x) = f'(v)$, $f(y) = 2$, and $f(u) = f(w) = 0$. If $f'(v) = 0$, then f' puts weight 2 on a neighbor of x or y, say x; let $f(y) = 2$ and $f(x) = f(u) = f(w) = 0$.

(c) If x and y are not opposite on C, then case (a) applies. Otherwise, form G' by contracting C into a single vertex v and adding a 3-cycle C' through v and two new vertices. An RDF f' of G' must put total weight at least 2 on $V(C')$. Let $f(x) = f(y) = 2$, put weight 0 on $V(C) - \{x, y\}$, and let $f(z) = f'(z)$ for $z \in V(G) - V(C)$.

In each case, $w(f) \leq w(f') + 2$. $\qquad \square$

A *spider* is a tree consisting of at least three paths having a common endpoint. The common endpoint is the only vertex of degree at least 3 in the spider and is its *branchpoint*. A spider is completely specified by listing the distances of the leaves from the branchpoint.

Lemma 4.2 *If G is an n-vertex spider with branchpoint v, then $\gamma_R(G) \leq 8n/11$ unless $d(v) = 3$ and the leaves have distances $(1, 3, 3)$ or $(2, 2, 3)$ from v. Among the remaining spiders, $\gamma_R(G) < 8n/11$ unless $d(v) = 4$ and the leaves have distances $(1, 3, 3, 3)$ or $(2, 2, 3, 3)$ from v, or $d(v) = 3$ and the leaf distances from v are obtained from $(1, 3, 3)$ or $(2, 2, 3)$ by adding 3 to one coordinate.*

Proof. Let l_i be the number of leaves at distance i from v. Suppose first that the longest path from v has length at most 3, so $n = 1 + l_1 + 2l_2 + 3l_3$. For any path of length 3 from v, f puts weight 2 on the penultimate vertex and weight 0 on the others.

If $l_1 = l_2 = 0$, then $l_3 \geq 3$. Complete the RDF f by $f(v) = 1$. Now $w(f) = 1 + 2l_3$, and $1 + 2l_3 < \frac{8}{11}(1 + 3l_3)$ when $l_3 \geq 2$.

If $l_1 = 0$ and $l_2 = 1$, then put weight 2 on the neighbor of v along the short path, and let $f(v) = 0$. Now $w(f) = 2 + 2l_3$, and $2 + 2l_3 < \frac{8}{11}(3 + 3l_3)$ when $l_3 \geq 0$.

Otherwise, let $f(v) = 2$ and put weight 1 on leaves at distance 2 from v to complete the RDF f. Now $w(f) = 2 + l_2 + 2l_3$. We seek $2 + l_2 + 2l_3 < \frac{8}{11}(1 + l_1 + 2l_2 + 3l_3)$, which is equivalent to $14 < 8l_1 + 5l_2 + 2l_3$. Since we have $l_1 + l_2 + l_3 \geq 3$ and $l_1 + l_2 \geq 1$ with equality in the latter only when $l_1 = 1$, the right side is at least 15 except in four cases. For $(l_1, l_2, l_3) \in \{(1, 0, 2), (0, 2, 1)\}$ the right side is 12, and we have $n = 8$ and $\gamma_R(G) = 6$. For $(l_1, l_2, l_3) \in \{(1, 0, 3), (0, 2, 2)\}$ the right side is 14, and we have $n = 11$ and $\gamma_R(G) = 8$.

With the spiders above as a basis, we now apply induction on n. We may assume that G has some path of length more than 3 from v. Let G' be the graph obtained from G by deleting three vertices from the end of a longest such path. Using weight 2 on the middle of those three vertices yields $w(G) \leq w(G') + 2$. Since $2/3 < 8/11$, the induction hypothesis yields $\gamma_R(G) < 8n/11$ unless G' is one of the two 8-vertex spiders that fail the bound. In this case, $n = 11$ and $\gamma_R(G) \leq 8$, so the desired ratio holds with equality. $\square$

A *thread* in a graph G is a trail whose internal vertices have degree 2 in G and whose endpoints do not have degree 2. If the endpoints of a thread are equal, then it forms a cycle having one vertex of degree greater than 2. In a connected graph with maximum degree at least 3, the threads decompose the edge set.

Theorem 4.3 *If G is a connected n-vertex graph with $\delta(G) \geq 2$ other than those shown below, then $\gamma_R(G) \leq 8n/11$.*
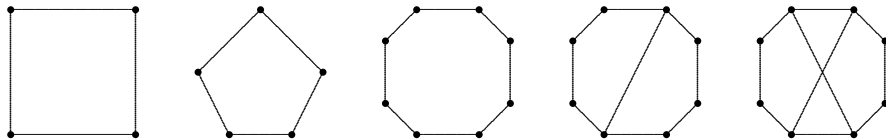
Proof. Note that $\gamma_R(C_4) = 3 > \frac{32}{11}$, $\gamma_R(C_5) = 4 > \frac{40}{11}$, and $\gamma_R(C_8) = 6 > \frac{64}{11}$. Also, one or two chords added to C_8 as shown above do not reduce γ_R. For each graph G shown above, $\frac{8|V(G)|}{11} < \gamma_R(G) \leq \frac{8|V(G)|}{11} + \frac{4}{11}$.

To prove the upper bound for all other graphs, we use induction on n. If G is a cycle, then the claim holds ($\gamma_R(C_7) = 5 < \frac{56}{11}$ and $\gamma_R(C_{11}) = 8$), so we may assume that $\Delta(G) \geq 3$. Our aim is to find a spanning subgraph of G in which one component G_1 is a spider to which we can apply Lemma **??**, and the remainder G_2 is a graph to which we can apply the induction hypothesis. First we use the induction hypothesis to restrict the structure of G.

Since $2/3 < 8/11$, Lemma **??**(a) allows us to assume that G has no induced path with at least three internal vertices.

Since deleting an edge cannot reduce γ_R, we may assume that every edge joining two vertices with degree at least 3 is a cut-edge. In particular, no cycle in G has a chord. If G has a cut-edge uv with endpoints of degree at least 3, then let H_u and H_v be the components of $G - uv$ containing u and v, respectively. Both H_u and H_v are edge-minimal connected graphs with minimum degree at least 2.

Let $\mathcal{C} = \{C_4, C_5, C_8\}$. If neither H_u nor H_v lies in $\mathcal{C}$, then the RDFs guaranteed for them by the induction hypothesis combine to form the desired RDF of G. If $H_u, H_v \in \mathcal{C}$, then in each case weight 2 on u permits saving one unit on H_v, so

$$\gamma_R(G) \leq \gamma_R(H_u) + \gamma_R(H_v) - 1 \leq \frac{8|V(H_u)| + 4}{11} + \frac{8|V(H_v)| + 4}{11} - 1 < \frac{8n}{11}.$$

Thus when G has a cut-edge uv with $d_G(u), d_G(v) \geq 3$, we may assume that exactly one of $\{H_u, H_v\}$ lies in $\mathcal{C}$.

Now let v be a vertex of degree at least 3 that is not in a member of $\mathcal{C}$ joined to the rest of G by a cut-edge. By the preceding paragraph, such a vertex exists; at least one end of every thread is suitable. We seek a subgraph G_1 consisting of $d(v)$ paths from v whose lengths do not equal 3, such that $\delta(G - V(G_1)) \geq 2$ and no component of $G - V(G_1)$ lies in $\mathcal{C}$. By Lemma **??** and the induction hypothesis, such a subgraph completes the proof.

Consider the threads emanating from v. If v lies on a cycle C whose other vertices have degree 2, then regardless of the length of C, it is possible to delete one edge e of C so that $C - e$ consists of two threads from v with neither having length 3.

All other threads from v lead to other vertices of degree at least 3. Let u be such a vertex, reached by a thread P. If u has degree 3 and lies on a cycle C whose other vertices have degree 2, then P reaches u via a cut-edge, and $C \in \mathcal{C}$ by our earlier remarks. Cutting an edge e of C incident to u leaves $P \cup C - e$ as a long thread leaving v; we put it in G_1.

Otherwise, we want to cut the last edge of P to obtain a short thread leaving v (by Lemma **??**(a), P has at most two internal vertices). However, cutting two threads that reach u from v could leave u with insufficient degree. By Lemma **??**(b,c), we may assume that exactly one thread P of length 2 and one thread P' of length 3 reach u from v.

If $d(u) \geq 4$, then we can cut each final edge. If $d(u) = 3$, then a third thread Q leaves u, ending at w. If w is not the end of another thread from v, or if $d(w) \geq 4$, then we can cut the last edge of P or P' (since their lengths differ) so that the resulting thread from v that is formed by cutting the end of Q incident to w does not have length 3.

If w is the end of one other thread from v in G and $d(w) = 3$, then we cut the last edge of P. The thread P' extends to reach w with length at least 3. When we cut the last edge of the other thread from v to w, the thread along P' and Q becomes even longer. The process can continue when v has large degree, yielding one long thread and many short threads.

If the process reaches some w' that is the end of two threads from v, and $d(w') = 3$, then cutting the edge reaching w' leaves a 5-cycle through v whose other vertices have degree 2 (the union of those two threads), and we can cut one edge of it to obtain two short threads.

In the remaining spanning subgraph, the component G_1 containing v is a union of $d(v)$ threads, none having length 3, and every other component has minimum degree at least 2 and is not one of the excluded subgraphs. As remarked above, Lemma **??** and the induction hypothesis now provide the desired RDF. $\square$

To characterize equality in Theorem **??**, we study its proof closely.

Theorem 4.4 *Let F be the graph of Figure **??**. Let G be a connected graph of order n with minimum degree at least 2. If $n \geq 9$, then $\gamma_R(G) = 8n/11$ if and only if*
(1) $n = 11$ and G is isomorphic to F plus a subset of one of $\{y_1y_3, y_1y_4, y_2y_3, y_2y_4\}$, $\{wz_1, y_1y_3, y_1y_4\}$, or $\{wz_1, wz_3, y_1y_3\}$ added as edges, or
(2) $n > 11$ and G consists of disjoint copies of the graphs F, $F + wz_1$, and $F + wz_1 + wz_2$ with additional edges connecting copies of w.
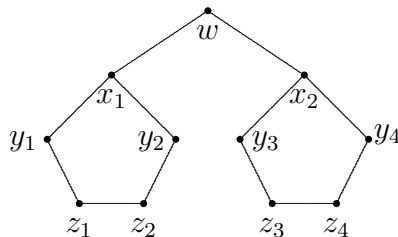

Figure 3: The graph F.

Proof. If G has the indicated form, then, regardless of the edges between copies of w, any RDF must put weight at least 8 on every copy of F, so $\gamma_R(G) \geq 8n/11$.

For the converse, let G be a graph achieving equality in Theorem **??**. Since $2/3 < 8/11$, G cannot contain a configuration as described in Lemma **??**. Also the deletion of any cut-edge joining vertices of degree at least 3 without leaving a component in $\mathcal{C}$ must leave components where equality holds.

Let G' be the subgraph resulting from such deletions (called G in Theorem **??**). Let v be a vertex of G' as chosen in that proof. Since equality holds for G', it must also hold for the subgraphs G_1 and $G' - V(G_1)$ obtained in the inductive proof.

A closer look at Lemma **??** characterizes the vectors of path lengths where $\gamma_R(G_1) = 8|V(G_1)|/11$ can hold. Since the proof of Theorem **??** extracts a graph G_1 in which no thread from v has length 3, equality requires the threads from v to have lengths 2, 2, and 6.

To obtain a thread of length 6 without obtaining a thread of length 1, we must have had $d(v) = 3$, and one thread from v reaches a cycle in $\mathcal{C}$. If $n = 11$, then the possibilities are as shown below, but the graph on the left has an RDF of weight 7. Inspection shows that the only graphs with Roman domination number 8 spanned by F are those claimed.
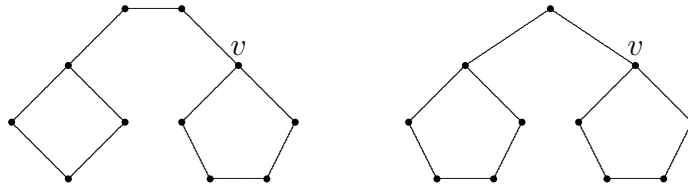
When $n > 11$, we claim that the endpoints of the threads of length 2 from v are still adjacent and have degree 2. If not, then they would have degree at least 3, and using one of them in place of v would yield a tree for G_1 that has a thread of length 1 (by cutting the edge of the thread to v). We would then have $\gamma_R(G') < 8n/11$.

We conclude that successively deleting edges of G with endpoints of degree at least 3, without introducing components in $\mathcal{C}$, yields a graph whose components are copies of F. Since there exist minimum weight RDFs of F putting weight 2 on any given vertex, and deletion of any vertex of F other than w leaves a subgraph where weight 7 suffices, every edge of G not contained among the vertices of a single copy of F joins copies of the central vertex of F.

If any edge of G connects the two 5-cycles in one copy F' of F, then since G is connected, the central vertex w' of F' has a neighbor in another copy of F that can be given weight 2. With w' protected, we can protect the rest of F' with weight 7 using the edge joining the two 5-cycles. This yields $\gamma_R(G) \leq 7 + 8(n - 11)/11 < 8n/11$. Hence no edges can be added between or within the copies of F other than those described in the statement. $\square$

5   2-Connected Graphs

In this section, we ask whether a connectivity threshold improves the upper bound on Roman domination number. As remarked in the Introduction, connectedness and minimum degree k force $\gamma_R(G) \leq c_k n$, where $c_k = O(\ln k/k)$. Perhaps requiring k-connectedness instead of minimum degree k will reduce the constant in the upper bound or even yield $\gamma_R(G) \leq c'_k n$ with $c'_k = o(c_k)$ as $k \to \infty$.

Conjecture 5.1 *If G is a 2-connected graph of order n, then $\gamma_R(G) \leq \lceil 2n/3 \rceil$.*

This bound is sharp if true, since $\gamma_R(C_n) = \lceil 2n/3 \rceil$ [?]. Conjecture ?? is equivalent to the same statement for *minimally 2-connected graphs*, which are the 2-connected graphs having no edge whose deletion leaves a 2-connected graph. We study minimally 2-connected graphs, using the following characterization.

Definition 5.2 *A* Level 0 graph *is a cycle of length at least 4. Two vertices x and y in a graph G are* compatible *in G if no x, y-path in G has a chord. A* Level k graph *is a graph obtained from graphs $G_1, \ldots, G_m$ (with $m \geq 2$), where each G_i is an edge or a Level j graph for some $j < k$, in the following way. In each G_i a compatible pair $\{x_i, y_i\}$ is selected, and the graphs are combined by identifying y_i with $x_{i+1} \pmod{m}$, for each i. If $m = 2$, then we do not allow G_1 or G_2 to be a single edge.*

Theorem 5.3 (Dirac [**?**]) *A graph is minimally 2-connected if and only if it is C_3 or is a Level k graph for some $k \geq 0$.*

We have remarked that Conjecture **??** holds for cycles, which include C_3 and all Level 0 graphs; we will prove it for Level 1 graphs.

Theorem 5.4 *If G contains a spanning Level 0 or Level 1 graph, then $\gamma_R(G) \leq \lceil 2n/3 \rceil$.*

Proof. Let $n = |V(G)|$; we use induction on n. In the basis step we include all graphs having spanning cycles. For the induction step, we may assume that G is a Level 1 graph. We may also assume by Lemma **??**(a) that G has no thread with at least three internal vertices, since contraction of an edge incident to a vertex of degree 2 preserves not only minimum degree 2 but also 2-connectedness.

Let $G_1, \ldots, G_m$ be the graphs used in the construction of G according to Definition **??**. Each G_i is an edge or is a cycle of length at least four. In the latter case, x_i and y_i are nonadjacent. By the exclusion of long threads, when G_i is a cycle the paths joining x_i and y_i on G_i have length 2 or 3, and also at most three consecutive subgraphs in the list are single edges. (If only one G_i is a cycle, then the contraction can produce a cycle with one chord, which is not minimally 2-connected but has a spanning Level 0 subgraph.)

Hence we may assume that each G_i is an edge or is a k-cycle with $k \in \{4, 5, 6\}$. Let $S = \{i \colon G_i \text{ is a cycle}\}$. For $i \in S$, let i' be the first index in the list $i+1, i+2, \ldots$ (modulo m) such that $i' \in S$. Let $l = i' - i \mod m$. We have restricted l to $\{0, 1, 2, 3\}$. For $i \in S$, let $U_i = \left(\bigcup_{j=i}^{i'} V(G_i) \right) - \{x_{i'}\}$. The sets U_i for $i \in S$ partition $V(G)$.

It suffices to specify an RDF f for G that puts weight at most $2|U_i|/3$ on each U_i for $i \in S$. First, let $f(x_i) = 2$ for all $i \in S$. Given $i \in S$, let k be the length of the cycle G_i, and define l as above; we have $|U_i| = k + l - 1$. We must still guard the subgraph H_i induced by $U_i - (N(x_i) \cup N(x_{i'}))$. Let w_i be the total weight we will place on U_i.

If $l = 0$, then H_i is empty and $w_i = 2$, which is small enough since $|U_i| = k - 1 \geq 3$. If $l = 1$, then $H_i = \overline{K}_{k-4}$; put weight 1 on each vertex of H_i. Now $w_i = k - 2$, which is small enough since $|U_i| = k \leq 6$. If $l = 2$, then $H_i = P_{k-3}$ (with center y_i); add weight $\lceil 2(k-3)/3 \rceil$ on y_i to guard H_i. Now $w_i = \lceil 2k/3 \rceil$ and $|U_i| = k + 1$, so w_i is small enough. Finally, if $l = 3$, then $H_i = K_{1,k-3}$ (with center y_i); put weight 2 on y_i to guard H_i. Now $w_i = 4$, which is small enough since $|U_i| = k + 2 \geq 6$. $\qquad \square$

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