Rainbow Spanning Trees in Abelian Groups

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May 12, 2020

Dedicated to Frederick Hoffman on the occasion of the 50^{th} Anniversary of the Southeastern International Conference on Combinatorics, Graph Theory & Computing

Abstract

Let $(A, +)$ be a finite Abelian group. Take the elements of A to be vertices of a complete graph and color the edge ab with $a+b$. A tree in A is *rainbow* colored provided all of its edges have different colors. In this paper we study conditions that regulate whether or not a given tree can be realized as a rainbow spanning subtree of an Abelian group of the same order. For example, let $C[h_1, \ldots, h_s]$ denote the caterpillar with s spine vertices and with h_i hairs on the ith spine vertex. We characterize, by means of divisibility conditions, when a caterpillar of type $C[k, \ell], C[k, 0, \ell]$ or of type $C[k, 0, 0, \ell]$ embeds as a rainbow spanning tree in a group of the same order. We also show that embeddability as a rainbow spanning tree is not a local condition. That is, given any tree T and sufficiently large non-cyclic group A, some trees of order $|A|$ that contain T as a subtree do embed as rainbow spanning trees in A, and some do not.

For non-Boolean groups A of order at most 20, we give a complete catalogue of all trees that fail to embed as rainbow spanning trees of A. We also show that all rainbow spanning trees in A can be obtained from the star with center 0 through a simple pivoting procedure.

1 Introduction

In this paper all graphs will be finite and simple, and all groups will be finite and Abelian. A group is *Boolean* if $x^2 = 1$ for all x. Such groups are isomorphic to $\mathbb{Z}_2^d$. With the exception of this introductory discussion, all groups in this paper will be non-Boolean — namely, they

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will contain at least one element of order greater than 2. The assignment of one color to each edge of a graph G is an *edge-coloring* of G. The coloring is *proper* if incident edges receive different colors and *rainbow*[1] if all edges receive different colors. One area of study of edge-colorings of graphs focuses on the following problem:

> Given a complete graph K_n whose edges have been colored and a graph G, does G have a rainbow embedding in that K_n? That is, does K_n have a subgraph isomorphic to G all of whose edges have different colors?

In anti-Ramsey theory [2] G is K_m for fixed m and a sufficiently large n is sought such that any proper edge-coloring of K_n contains a rainbow colored $G = K_m$. In a similar vein, Lars Andersen [1] has shown that in any properly edge-colored complete graph of order n, there is a Hamiltonian cycle whose edges use at least $n - \sqrt{n}$ distinct colors. In these cases, the edge-coloring of K_n has no particular structure other than being proper. The cases where the edge-coloring arises from some specific geometric or algebraic structure have also been studied.

A coloring can be determined geometrically by a finite set of points X in the plane $\mathbb{R}^2$ as follows. Form the complete graph with X as vertex set. Color the edge pq with the slope of the line from p to q (vertical lines get color ∞). In this setting a rainbow spanning tree was called a *direction tree* [9, 10] — that is, a spanning tree in which the slopes of all the edges are distinct. This "slope coloring" fails to be proper when there is collinearity among the points. Nonetheless, a direction tree does exist if not all points of X are collinear [10]. (This result was extended to an abstract setting in [8].) In [9, 10], it was conjectured that if no 3 points of X are collinear, then X possesses a direction path. This was proved by Kleitman and Pinchasi [14], who showed that the conjecture still holds when "path" is replaced by an arbitrary caterpillar.

The edge-colorings studied in this paper arise from finite Abelian groups. Formally, we associate with each finite Abelian group A an abstract edge-colored complete graph K_A. The vertices of K_A lie in one-to-one correspondence with the elements of A, via $v_a \leftrightarrow a$, and the edge $v_a v_b$ has color $a + b$.

The first result on rainbow spanning trees in K_A is due to Maamoun and Meyniel [15], who, in answer to a question of Gena Hahn, showed that the path cannot occur as a rainbow spanning tree of the Boolean group $\mathbb{Z}_2^d$ for $d > 1$. Incidentally, in reference to the result of Lars Andersen mentioned above, with $n = 2^d$, the Boolean group $\mathbb{Z}_2^d$ cannot have 0 as an edge label. Thus $n - 1$ is a bound on the size of the largest possible rainbow cycle. Rainbow spanning trees in general were introduced in [11] in terms of labelings and studied for Boolean groups by Zheng [20] (see also [12]).

The description of rainbow spanning trees of K_A in terms of labelings is as follows. Given a graph G, a *labeling* of G by an Abelian group A is a map $\lambda : V(G) \to A$. The *edge-coloring* λ^* *induced by* λ is given by $\lambda^*(xy) = \lambda(x) + \lambda(y)$. For clarity, we generally refer to images under λ as *labels* and images under λ^* as *colors*. That is, vertices have labels, while edges

[1] The terms *heterochrome*, *polychrome*, *antihomogeneous* and *totally multicolored* have also been used.

have colors. A labeling λ of a graph G is *rainbow over* A provided λ and λ^* are both injective. A graph admits a rainbow labeling over A is sometimes said to be A-*rainbow*.

Many related types of labelings have been studied in the literature (see [5] for a thorough survey). Those most closely connected to the present paper are harmonious labelings and cordial labelings. In 1980 Graham and Sloane [6] defined a *harmonious labeling* of a graph G with $m = |E(G)| = |A|$ to be an labeling over the cyclic group $\mathbb{Z}_m$ in which all vertex labels are distinct, as are all edge colors. Harmonious labelings were later generalized to arbitrary Abelian groups by Beals, Gallian, Headley, and Jungreis [3]. They gave necessary and sufficient conditions for a cycle to have a harmonious labeling over A (see also [18, 19]). Note that when G is a cycle, A-harmonious labelings are equivalent to rainbow labelings over A.

A *cordial labeling* over A of a graph G is an labeling λ of G over A such that, for all $a, b \in A$, we have $||\lambda^{-1}(a)| - |\lambda^{-1}(b)|| \leq 1$ and $||(\lambda^*)^{-1}(a)| - |(\lambda^*)^{-1}(b)|| \leq 1$. In general, an element of A may appear multiple times in a cordial labeling as either a vertex label or as an edge label. The cordiality condition says that each vertex label appears nearly the same number of times, as does each edge color. Cordial labelings were first introduced by Cahit [4] over $A = \mathbb{Z}_2$. Hovey [7] generalized the concept to arbitrary Abelian groups.

Since the vertex labels need to be "balanced", when G is a tree of order $|A|$, each element of A must appear exactly once as a vertex label and, with one exception, exactly once as an edge label. Thus, when the order of the group is the same as the order of the tree, a cordial labeling over A is exactly the same as a rainbow labeling over A. Hovey [7] showed that all n-vertex caterpillars are $\mathbb{Z}_n$-cordial (and hence $\mathbb{Z}_n$-rainbow), and conjectured that the same is true of all n-vertex trees.

In the present paper, the emphasis is on finding rainbow spanning trees of K_A. A tree T *embeds as a rainbow spanning tree (RST)* of K_A provided T is isomorphic to a rainbow spanning subtree of K_A. This is equivalent to saying that T has an *RST labeling over* A: a rainbow labeling over A where $|V(T)| = |A|$. Both perspectives will be used, as appropriate, in this paper. We begin with general results about arbitrary trees and end with specific results on several classes of caterpillars. Section 3 contains two opposing results on embedding an arbitrary tree T as a subtree of a rainbow spanning tree (RST) over an Abelian group. We show in Theorem 3.2 that if A is large enough, then some RST of A contains T as a subtree. On the other hand, Theorem 3.3 shows that that if A is large enough and *not cyclic*, then there is a tree T^* of order $|A|$ containing T as a subtree that has no rainbow labeling over A. This shows that the existence of rainbow labelings cannot be determined by local conditions.

Moreover, it shows that Hovey's conjecture on the $\mathbb{Z}_n$-cordiality of all trees cannot be extended to A-cordiality for any non-cyclic A. The section concludes with a construction through which rainbow spanning trees of K_B and K_C can be combined to yield a rainbow spanning tree of $K_{B\times C}$.

In Section 4 we give a procedure to transform one rainbow spanning tree of K_A into another, and show that this procedure can be used to produce all possible rainbow spanning trees. In Section 5 we examine several classes of caterpillars and, for each class, characterize those having rainbow labelings over A. Finally, in Section 6 we provide a catalogue, for all non-Boolean Abelian groups A of order at most 20, of all trees of order $|A|$ that fail to be A-rainbow.

2 The VE-equation

We begin the general presentation with a useful technique, introduced in [11] and utilized also in [12, 20]. Suppose tree T has a rainbow labeling λ over a non-Boolean Abelian group A, where $|V(T)| = |A|$. Let λ^* denote the edge-coloring induced by λ. Since 0 can now appear as an edge label, there are fully $|A|$ possible edge labels, but only $|A|-1$ edges in T. Thus exactly one color, which will be designated as the *free color* f, is not used as an edge label. Since the label on v contributes $\lambda(v)$ to the color of each incident edge, we have

$$\sum_{v\in V(T)} \deg(v)\lambda(v) = \sum_{e\in E(T)} \lambda^*(e). \qquad (*)$$

Letting $s = \sum_{a\in A} a$, we have both $s = \sum_v \lambda(v)$ and $s = f + \sum_e \lambda^*(e)$. Subtracting s from both sides of (*) yields the *vertex-edge equation* (or *VE-equation*):

$$\sum_{v\in V(T)} [\deg(v) - 1]\lambda(v) \;=\; -f.$$

The VE-equation yields a constraint that is often very useful in ruling out the existence of a rainbow labeling. The *characteristic*[2] of a group A is the least common multiple of the orders of its elements. In an Abelian group the characteristic is the smallest positive integer m such that $ma = 0$ for all $a \in A$.

Theorem 2.1. *Let A be an Abelian group with order n and characteristic m, and let T be an n-vertex tree. If T has adjacent vertices u and v such that*

- $\deg(u) \equiv \deg(v) \equiv 0 \pmod{m}$,
- $\deg(x) \equiv 1 \pmod{m}$ *for all* $x \in V(T) \setminus \{u, v\}$, *and*
- $uv \in E(T)$,

then T is not A-rainbow.

Proof. Suppose to the contrary that T has an RST labeling λ over A, and let f be the free color. Since $\deg(x) \equiv 1 \pmod{m}$ for all $x \in V(T) \setminus \{u, v\}$, and since A has characteristic m, we have

$$\sum_{x\in V(T)\setminus\{u,v\}} [\deg(x) - 1]\lambda(x) = 0.$$

[2] Sometimes also called the *exponent* of the group.

Thus the VE-equation yields

$$-\lambda(u) - \lambda(v) = \sum_{x \in V(T)} [\deg(x) - 1]\lambda(x) = -f,$$

so $f = \lambda(u)+\lambda(v)$. Now the observation that color $\lambda(u)+\lambda(v)$ appears on edge uv contradicts the choice of f as the free color. ■

3 Embedding and Extending

In this section we briefly discuss the existence and non-existence of rainbow labelings for general trees. We begin by showing that, for any Abelian group A, every tree whose order is sufficiently small relative to $|A|$ embeds in an RST of A. We split this proof into two parts. (The lemma below holds in any properly edge-colored complete graph.)

Lemma 3.1. *For any Abelian group A, every rainbow subtree of K_A can be extended to a rainbow spanning tree.*

Proof. Let T be a rainbow subtree of K_A, let $n = |A|$, and let $k = |V(T)|$. We proceed by induction on $n - k$. If $n - k = 0$, then T is already an RST of K_A, and there is nothing to prove. Otherwise, it suffices to show that T can be extended to a larger rainbow subtree of K_A. Let v_a be any vertex of K_A not belonging to T. Consider the edges joining v_a to the vertices of T. There are k such edges, each having a different color; since only $k - 1$ colors appear on the edges of T, some edge from v_a to T has a color not used on T. Joining v_a to T with this edge yields a larger rainbow subtree of K_A. ■

Theorem 3.2. *If A is an Abelian group and T is a tree with $|A| \geq 2\,|V(T)| - 2$, then there exists a rainbow spanning tree S of K_A such that T is isomorphic to a subtree of S.*

Proof. Let the vertices of T be $t_1, \ldots, t_k$, ordered so that each vertex other than t_1 has exactly one earlier neighbor. (For example, it suffices to take a breadth-first ordering starting from an arbitrary root.) We map the vertices of T onto vertices of K_A in order, so that at all times the subgraph of T embedded so far is rainbow.

Begin by mapping t_1 onto an arbitrary vertex of K_A. Next suppose we have already embedded $t_1, \ldots, t_i$, and now want to embed t_{i+1}. By the choice of ordering, t_{i+1} has one earlier neighbor, say t_j. The vertex t_j has already been mapped onto some vertex of K_A, say v_a; we need to map t_{i+1} onto some unused vertex v_b such that $a + b$ does not appear on any edge embedded thus far. Since $|A| \geq 2k - 2$, at least $k - 1$ elements of A remain unused on vertices of T. Each yields a different sum with a, and at most $k - 2$ sums already appear on edges of T, so there is at least one valid choice for v_b.

This results in an isomorphic copy T^* of T in K_A that is rainbow edge-colered. By Lemma 3.1, T^* can be extended to the desired S. ■

Thus for any Abelian group A and tree T, if $|A| \geq 2|V(T)|-2$, then T is contained within an A-rainbow tree. We next show that every tree with sufficiently few vertices (relative to $|A|$) is contained within a tree of order $|A|$ that is not A-rainbow. Hence one cannot establish "forbidden subtree" conditions for the existence or non-existence of a rainbow labeling.

Theorem 3.3. *Let A be an Abelian group with order n and characteristic m and let T be any tree. If $|V(T)| \geq 2$ and $n \geq m|V(T)|$, then there is an n-vertex tree T^* such that $T \subseteq T^*$ and T^* is not A-rainbow.*

Proof. We construct T^* from T by attaching pendant leaves to the vertices of T, with the aim of invoking Theorem 2.1. Fix any adjacent vertices u and v in T. To each of these vertices, attach just enough pendant leaves to bring the degree up to a multiple of m; to every other vertex, attach just enough pendant leaves to make the degree congruent to 1 $\pmod m$.

Since at most $m-1$ pendant vertices were attached to each vertex of T, the size of the resulting tree cannot exceed $m|V(T)|$. Moreover, u and v each have degree congruent to 0 $\pmod m$, while every other vertex has degree congruent to $1 \pmod m$. (In particular, each new pendant vertex has degree 1.) Let k denote the number of vertices in the tree. Since the tree has $k-2$ vertices with degrees congruent to $1 \pmod m$ and two with degrees congruent to $0 \pmod m$, the sum of the degrees is congruent to $k-2 \pmod m$. On the other hand, the sum of the degrees is exactly twice the number of edges, which is $2k-2$. We conclude that $k \equiv 0 \pmod m$; since also $n \equiv 0 \pmod m$, we have $n-k \equiv 0 \pmod m$. Complete the construction by adding $n-k$ additional pendant leaves to u; this does not change the fact that the degree of u is congruent to $0 \pmod m$.

The resulting tree has exactly n vertices. It contains T and, by Theorem 2.1, it does not have a rainbow labeling over A. ∎

Note that if A is cyclic, then $m = n = |A|$, so T cannot satisfy the requirement of Theorem 3.3. This should come as no surprise in light of Hovey's conjecture on the existence of $\mathbb{Z}_k$-cordial (and hence rainbow) labelings of trees. However, whenever A is non-cyclic, the proposition applies with $T = K_2$. Hence we have

Corollary 3.4. *For every non-cyclic Abelian group of order n, there exists an n-vertex tree that is not A-rainbow.*

There are not many general constructions for rainbow labelings, contrary to the expectation that almost all trees do have rainbow labelings. As promised at the end of the introduction, we conclude this section with a construction for rainbow labelings over products of Abelian groups that places only mild restrictions on the trees from the factors. Let $A = B \times C$, where B is an Abelian group of odd order and C is an arbitrary Abelian group. Let T be a tree with B-rainbow labeling τ, where $|V(T)| = |B|$. For each vertex v of T, let S_v be a tree with C-rainbow labeling σ_v, where $|V(S_v)| = |C|$; moreover, we require that the free color under each σ_v is zero[3]. We assume that the S_v are vertex-disjoint.

[3]Such labelings are sometimes called *A-elegant* [3].

Create an amalgamated tree R with vertex set $\bigcup\{V(S_v) : v \in V(T)\}$ as follows: for all vertices v in T, identify v with the vertex of S_v that receives label 0 under σ_v. Thus R has edges going "along" T and, for each $v \in V(T)$, edges "dropping down" through S_v.

Construct a labeling λ of R as follows: for vertex u in S_v, define $\lambda(u) = (\tau(v), \sigma_v(u))$. A vertex v in R arising from a vertex in T has label $(\tau(v), 0)$ by our amalgamation requirement. Every vertex in S_v has $\tau(v)$ as its first coordinate. Since τ takes on all values in B on the vertices of T and each σ_v takes on all values in C on the vertices of S_v, it follows that λ takes on all labels in $A = B \times C$. Since 0 is the free color in each S_v under σ_v, the set of colors induced by σ_v on the edges of S_v is $C \setminus \{0\}$. Now for each c in $C \setminus \{0\}$ and each S_v, there is some edge xy in S_v such that $\sigma_v(xy) = c$. Looking at this in all of R, the vertices x and y have coordinates $\lambda(x) = (\tau(v), \sigma_v(x))$ and $\lambda(y) = (\tau(v), \sigma_v(y))$. The color of xy is thus $(2\tau(v), \sigma_v(x) + \sigma_v(y))$, which equals $(2\tau(v), c)$. Since B has odd order, the map $g \to 2g$ is a bijection of B onto B. Hence as $\tau(v)$ ranges over B, the values $(2\tau(v), c)$ fill out the entire coset $B \times \{c\}$. It follows that the edges "dropping down" in R have distinct edge labels in R.

By choice of the amalgamation vertices, the vertices of R corresponding to the vertices of T take on the labels $(g, 0)$ as g ranges through B. Thus since the vertices coming from T all have second coordinate 0, the edge colors in T all have second coordinate 0, and lie in the subgroup $B \times \{0\}$ of A. Hence the edge colors in R are distinct.

4 Pivoting

In this section we study a procedure for transforming one rainbow spanning tree (henceforth *RST*) of K_A into another. Given such a tree T, consider any edge e in K_A of the free color. The graph $T+e$, viewed as a subgraph of K_A, has exactly one edge of each color and contains exactly one cycle. Deleting any edge in the cycle yields another RST of K_A. We call this operation — the addition of e and the removal of some edge in the cycle that arises — a *pivot*. Note that pivoting is a form of basis exchange in the graphic matroid on K_A but with the constraint imposed by the edge-coloring. Without the constraint, moving from one spanning tree to another is easy. With the color constraint it is more challenging.

Pivoting will occasionally prove convenient throughout the paper. In this section we show that, if A is not Boolean, then one can move from any RST of K_A to any other via a series of pivots. Since the inverse of any pivot is itself a pivot, it suffices to show that the spanning star centered at v_0 can be reached via pivots from any initial RST.

An *involution* in A is an element with order 2. Recall that a *Boolean group* is an Abelian group in which every non-identity element is an involution; such groups have the form $\mathbb{Z}_2^d$. In an Abelian group, the sum of distinct involutions is again an involution.

Theorem 4.1. *If A is a non-Boolean Abelian group and T is a rainbow spanning tree of K_A, then some sequence of pivots converts T into the spanning star of K_A centered at v_0.*

Proof. Let T be an RST of K_A. Let $\mathscr{R}$ be the set of all trees that can be reached from T by a sequence of pivots, and let $\mathscr{M}$ be the set of those trees in $\mathscr{R}$ that maximize the degree of

v_0. We will show that $\mathscr{M}$ consists solely of the spanning star centered at v_0. Let I be the set of involutions in A, and let J be the set of elements with order at least 3. Since A is not Boolean, J is nonempty.

Claim (1) For every RST in $\mathscr{M}$, the free color is 0.

Suppose S is some RST reachable from T by a sequence of pivots. If the free color of S is some nonzero f, then pivot by adding edge v_0v_f and removing an edge not incident to v_0 in the resulting cycle. This increases the degree of v_0, so $S \notin \mathscr{M}$.

Claim (2) For every RST in $\mathscr{M}$, the vertex v_0 is adjacent to all v_a with $a \in J$.

Suppose the claim fails for some $S \in \mathscr{M}$ and some $a \in J$. By Claim (1) we know that the free color in S is 0. Since a is in J, we have $a \neq -a$, so $v_a v_{-a}$ is an edge of K_A. Since this edge has color 0, it is not in S. Pivot by adding edge $v_a v_{-a}$ and removing the other edge e incident to v_a in the resulting cycle. Since we remove no edges incident to v_0, its degree is unchanged. Moreover, e does not have color 0, since 0 was the free color initially. Thus, after this pivot the new RST is again in $\mathscr{M}$ and the new free color is nonzero. This contradicts Claim (1).

This shows that v_0 is adjacent to all vertices except possibly to some of those corresponding to involutions. To finish, we must now attend to the involutions.

Claim (3) For every S in $\mathscr{M}$, if v_x is not adjacent to v_0 in S for some nonzero x, and v_y is any neighbor of v_x in S, then y is an involution.

By Claim (2), x must be an involution. Since x is an involution, $x + y = 0$ would imply $y = -x = x$, contrary to v_y being a neighbor of v_x. Hence $v_{x+y} \neq v_0$. Moreover, since edge $v_x v_y$ has color $x + y$, this color appears on no edges incident to v_0. That is, v_{x+y} is neither equal to nor adjacent to v_0. Thus $x+y$ must be an involution by Claim (2). Since the sum of distinct involutions in an Abelian group is again an involution, it follows that $y = (y+x)+x$ is an involution.

The next claim follows immediately from Claims (2) and (3).

Claim (4) For every S in $\mathscr{M}$, vertex v_a is a pendant leaf at v_0 for all $a \in J$.

We have now arrived at the final step in showing that $\mathscr{M}$ contains only the star at v_0. Fix $S \in \mathscr{M}$. If S is not the star, then some v_g is not adjacent to v_0. By Claim (2), g must be an involution. Viewing the tree as rooted at v_0, let v_h be the parent of v_g. Since v_g is not adjacent to v_0, Claim (3) implies that h must also be an involution.

Select a fixed but arbitrary $a \in J$. Since $-a$ has the same order as a, we also have $-a \in J$. Therefore v_a and v_{-a} are both adjacent to v_0 in S, so $v_a v_{-a}$ is not an edge of S. Hence we can pivot by adding $v_a v_{-a}$ and removing $v_a v_0$. This produces a rainbow spanning tree S_1 with the following properties:

(P1) v_t is adjacent to v_0 for all $t \in J \setminus \{a\}$;

(P2) the degree of v_0 is one less than the degree of v_0 in S;

(P3) the free color is a.

Now $2(a + g) = 2a + 2g = 2a \neq 0$ since g is an involution but $a \in J$. Thus $a + g \in J$. Since $a + g \neq a$, it follows from (P1) that v_{a+g} is adjacent to v_0 in S_1. The edge $v_{a+g}v_g$ has color $a + g + g$, which is a, the free color by (P3). Hence we can pivot on $v_{a+g}v_g$. By Claim (4) v_{a+g} is a pendant leaf at v_0 in S and hence remains so in S_1. Thus the cycle created by adding $v_{a+g}v_g$ passes through v_0 and hence also through v_h. To finish the pivot, delete the edge v_gv_h to produce a new tree S_2. This pivot does not change the neighborhood of v_0, so (P1) and (P2) remain true. The free color changes to $g + h$.

Since $g + h$ is the free color, v_{g+h} cannot be adjacent to v_0. Now pivot to obtain S_3 by adding the edge $v_{g+h}v_0$ and discarding an edge not incident to v_0 in the resulting cycle. This increases the degree of v_0 by 1, so that it now equals the degree of v_0 in S. Thus $S_3 \in \mathscr{M}$, but this contradicts Claim 2 since $a \in J$ but v_a is not adjacent to v_0 in S_3. ■

5 Caterpillars

A *caterpillar* is a tree obtained by adding pendant edges (*hairs*) to a path (*the spine*). Following [13], we denote by $C[h_1, \ldots, h_s]$ a caterpillar with s spine vertices and h_i hairs incident to the ith vertex of the spine. Note that there are multiple ways to denote a single caterpillar, due to symmetry and flexibility in choosing the spine. For example, the path P_3 is simultaneously $C[2]$, $C[0, 1]$, $C[1, 0]$, and $C[0, 0, 0]$.

In this section we explore the existence of rainbow labelings for caterpillars over various Abelian groups.

Hovey's conjecture [7] that all trees are $\mathbb{Z}_k$-cordial for all k would imply that every n-vertex tree is $\mathbb{Z}_n$-rainbow. This conjecture remains open for general trees. However, Hovey showed that all n-vertex caterpillars are $\mathbb{Z}_n$-cordial, and hence $\mathbb{Z}_n$-rainbow. This also follows from the result of Kleitman and Pinchasi [14] on rainbow caterpillars in slope colored n-gons. Indeed, if the vertices of a regular n-gon are cyclically labeled $v_0, \ldots, v_{n-1}$, it is easy to see that the segments v_iv_j and v_kv_m are parallel iff $i + j \equiv k + m \pmod{n}$, so a tree has a rainbow labeling over $\mathbb{Z}_n$ iff it can be represented as a direction tree in the regular n-gon.

Theorem 5.1. *[7, 14] Every n-vertex caterpillar has a rainbow labeling over $\mathbb{Z}_n$.*

Although all n-vertex caterpillars have rainbow labelings over $\mathbb{Z}_n$, they need not have rainbow labelings over all Abelian groups of order n. We next present results characterizing, for certain special families of caterpillars, those caterpillars having rainbow labelings.

We first present a sufficient condition for the existence of a rainbow labeling. Our argument makes use of the following concept from [11]. Given a tree T and an Abelian group A, a *closed labeling* of T over A is a labeling of the vertices of T with elements of A such that each label appears on at most one vertex, each color in the induced edge-coloring appears on at most one edge, and each color on an edge also appears as a label on a vertex. Note that a closed labeling of T using all labels in A is a rainbow labeling and conversely.

In what follows, we use the notation $N(v)$ to denote the *open neighborhood* of a vertex v, i.e. the set of all neighbors of v; similarly, $N[v]$ denotes the *closed neighborhood* of v, i.e. $N(v) \cup \{v\}$.

Theorem 5.2. *Let A be an Abelian group of order n having an element of order m. Let T be an n-vertex caterpillar such that $T = C[h_1, \ldots, h_k]$ and let $r_1, \ldots, r_k$ be the residues of $h_1, \ldots, h_k$ modulo m. If $r_1 + \ldots + r_k + k = m$, then T has a rainbow labeling over A.*

Proof. Let a be an element of order m in A. The cosets of $\langle a \rangle$ partition A; each coset has size m, so there are n/m of them. Let $q = n/m$ and let the cosets of $\langle a \rangle$ be $S_1, \ldots, S_q$, with $S_q = \langle a \rangle$.

Let $T^* = C[r_1, \ldots, r_k]$. Since $|V(T^*)| = m$, Theorem 5.1 implies that T^* is $\mathbb{Z}_m$-rainbow. Since $\langle a \rangle$ is isomorphic to $\mathbb{Z}_m$, a $\mathbb{Z}_m$-rainbow labeling of T^* yields a closed labeling of T^* using only those labels in S_q.

We extend this closed labeling of T^* to a rainbow labeling of T over A. Let $v_1, \ldots, v_k$ be the vertices of the spine of T, with v_i incident to h_i hairs. In total $(q-1)m$ vertices remain unlabeled, each a pendant leaf; the $(q-1)m$ unused labels are precisely those in $S_1, \ldots, S_{q-1}$. For each i, the definition of r_i implies $h_i = p_i m + r_i$ for some p_i. Distribute the cosets $S_1, \ldots, S_{q-1}$ among the vertices of the spine, with vertex v_i receiving p_i different cosets. Label the $p_i m$ unlabeled pendant leaves at v_i with the $p_i m$ elements in its assigned cosets. Since the label on v_i belongs to $\langle a \rangle$, the set of new colors induced on the hairs at v_i is precisely the union of the cosets assigned to v_i. Thus we have a closed labeling; since all vertices have been labeled, the labeling is in fact A-rainbow. ■

We next study several classes of "small" caterpillars. For each class, Theorems 5.2 and 2.1 facilitate simple necessary and sufficient conditions for the existence of rainbow labelings over arbitrary Abelian groups. We begin with caterpillars of the form $C[k, \ell]$, also known as *double-stars*.

Theorem 5.3. *Let A be an Abelian group with order n and characteristic m. An n-vertex caterpillar T of the form $C[k, \ell]$ has a rainbow labeling over A if and only if $k \not\equiv -1 \pmod{m}$.*

Proof. Let the spine of T consist of vertices u and v, which are incident to k and ℓ hairs respectively.

First suppose $k \equiv -1 \pmod{m}$; note that this implies $\deg(u) \equiv 0 \pmod{m}$. Since $N(u)$ and $N(v)$ partition $V(T)$, we have $n = \deg(u) + \deg(v)$. Now $n \equiv 0 \equiv \deg(u) \pmod{m}$, so also $\deg(v) \equiv 0 \pmod{m}$. Moreover, each vertex not on the spine is a leaf, so $\deg(x) = 1$ for all $x \in V(T) \setminus \{u, v\}$. It now follows from Theorem 2.1 that T does not have a rainbow labeling over A.

Now suppose $k \not\equiv -1 \pmod{m}$. As shown above, $\deg(u) + \deg(v) = n \equiv 0 \pmod{m}$, so $k + \ell \equiv -2 \pmod{m}$, and consequently $\ell \not\equiv -1 \pmod{m}$. Let r and s be the residues modulo m of k and ℓ, respectively. We have shown that neither r nor s equals $m - 1$. Since $k + \ell \equiv -2 \pmod{m}$ and $0 \le r + s \le 2m - 4$, we must have $r + s = m - 2$. Now T is A-rainbow by Theorem 5.2. ■

We next consider caterpillars of the form $C[k, 0, \ell]$.

Theorem 5.4. *Let A be an Abelian group with order n and characteristic m. An n-vertex caterpillar of the form $C[k, 0, \ell]$ has a rainbow labeling over A if and only if $k \not\equiv -1 \pmod{m}$ and $\ell \not\equiv -1 \pmod{m}$.*

Proof. Let T be a caterpillar with spine vertices u, v, and w, which are incident to k, 0, and ℓ hairs, respectively. Suppose $k \equiv -1 \pmod m$ and suppose, contrary to the assertion, that T has an A-rainbow labeling. Between them, u and w are incident to all edges in T. Viewing T as a rainbow subgraph of K_A, the edge uw does not belong to T, but is incident to all edges in T. Since each color appears on at most one edge incident to each vertex, the color on uw appears on no edges of T, and hence must be the free color. Pivot, adding uw and removing uv. This yields a rainbow copy of $C[k, \ell+1]$, which is impossible by Theorem 5.3. A symmetric argument applies when $l \equiv -1 \pmod m$.

Next suppose $k \not\equiv -1 \pmod m$ and $\ell \not\equiv -1 \pmod m$; we proceed as in the proof of Theorem 5.3. Let r and s be the residues modulo m of k and ℓ, respectively. By assumption, neither r nor s equals $m-1$. Since $k + \ell = n - 3 \equiv -3 \pmod m$ and $0 \le r + s \le 2m - 4$, we must have $r + s = m - 3$. It now follows from Theorem 5.2 that T has a rainbow labeling over A. ■

Finally, we consider caterpillars of the form $C[k, 0, 0, \ell]$. We begin with one particular family of such caterpillars.

Theorem 5.5. *If $A = (\mathbb{Z}_p)^k$ where $k \ge 2$ and p is an odd prime, then the caterpillar $C[p^k - p - 1, 0, 0, p - 3]$ does not have a rainbow labeling over A.*

Proof. Let $n = p^k$ and let T be a caterpillar with spine vertices u, v, w, and z which are incident to $n - p - 1$, 0, 0, and $p - 3$ hairs, respectively. Suppose, contrary to the assertion, that T has an rainbow labeling λ over A. We may assume without loss of generality that z has label 0, since otherwise we may obtain a rainbow labeling with this property by subtracting $\lambda(z)$ from all vertex labels.

Let $a = \lambda(u)$ and let $q = p^{k-1}$. Note that a has order p, since $A = (\mathbb{Z}_p)^k$ and $a \ne 0$. Let Λ denote the set of labels that appear on neighbors of u and are not in $\langle a \rangle$. We claim $\Lambda + a = \Lambda$. To see this, let x be a neighbor of u with $\lambda(x) \notin \langle a \rangle$. The color of edge xu is $\lambda(x) + a$, which appears also as a label on some vertex y. Since $\lambda(x) \notin \langle a \rangle$, it follows that $\lambda(x) + a \ne 0$, so $y \ne z$. If y is adjacent to z, then $\lambda(x) + a = \lambda(y) = \lambda(y) + \lambda(z)$, so $\lambda(y)$ occurs on two edges, ux and yz. This is impossible, so y is not adjacent to z. Since 0 is the label on z, it is not the label on x. Hence $\lambda(y) = \lambda(x) + \lambda(u) \ne \lambda(u)$, so $y \ne u$. Since $N[u]$ and $N[z]$ partition $V(T)$, it now follows that $y \in N(u)$. If $\lambda(y)$ were in $\langle a \rangle$, then $\lambda(x)$ (which is $\lambda(y) - a$) would also be in $\langle a \rangle$, contrary to the choice of x. Thus $\lambda(x) + a$ is again in Λ.

Hence Λ is a union of cosets of $\langle a \rangle$, so $|\Lambda|$ is a multiple of p. The degree of u is $n - p$. A priori, some neighbors of u could have labels in $\langle a \rangle$. The number of such neighbors is at most $p - 1$, since 0 appears on z and cannot also appear on a neighbor of u. Therefore, $n - 2p + 1 = n - p - (p - 1) \le |\Lambda| \le n - p$. The only multiple of p in this range is $n - p$, so $|\Lambda| = n - p$. That is, Λ is precisely the set of labels on neighbors of u. The set of colors on edges incident to u is $\Lambda + a$ which, as we have shown, is Λ.

The labels in $\langle a \rangle$ must occur on u and the vertices in $N[z]$. Since $w \in N[z]$, it follows that $\lambda(w) \in \langle a \rangle$. Now $\lambda(v) \in \Lambda$, so the color $\lambda(v) + \lambda(w)$ is also in Λ since Λ is a union of cosets of $\langle a \rangle$. Thus the color $\lambda(v) + \lambda(w)$ both on vw and on some edge incident to u, a contradiction. ■

Using the VE-equation and techniques similar to those in the proofs of Theorems 5.4 and 5.5, it is possible to obtain a general result for caterpillars of the form $C[k, 0, 0, \ell]$.

Theorem 5.6. *Let A be an Abelian group with order n and characteristic m. An n-vertex caterpillar of the form $C[k, 0, 0, \ell]$ has a rainbow labeling over A if and only if*
(a) $k \not\equiv -2 \pmod{m}$, and
(b) one of the following holds:

$$\text{(i) } m \text{ is not prime,} \quad \text{or} \quad \text{(ii) } k \neq m-3 \text{ and } \ell \neq m-3.$$

Proof. Let $T = C[k, 0, 0, \ell]$, and let it consist of vertices u, v, w, and z, which are incident to k, 0, 0, and ℓ hairs respectively.

Suppose condition (a) fails, but T is A-rainbow. View T as a rainbow subgraph of K_A. The edge uz does not belong to T, but it is incident to all edges of T except vw. Hence there are only two possibilities for the color on uz: either it is the free color, or it agrees with the color on vw. In either case, removing vw and adding uz yields a rainbow copy of $C[k+1, \ell+1]$, contradicting Theorem 5.3.

Note that $\ell = m-3$ is equivalent to $k = n-m-1$ since $n = k+\ell+4$. Similarly, $k = m-3$ is equivalent to $\ell = n-m-1$. Thus if (b) fails, then T is not A-rainbow by Lemma 5.5.

Now suppose conditions (a) and (b) both hold. Since $k+\ell+4 = n \equiv 0 \pmod{m}$, we see that also $\ell \not\equiv -2 \pmod{m}$. Let r and s be the residues modulo m of k and ℓ, respectively. Now $0 \le r+s \le 2m-2$. Since $k+\ell \equiv -4 \pmod{m}$, we have $r+s \in \{m-4, 2m-4\}$. If $r+s = m-4$, then Theorem 5.2 implies that T is A-rainbow. Otherwise, we produce an A-rainbow labeling of T using a different argument.

Suppose $r+s = 2m-4$. Since $r \neq m-2$ and $s \neq m-2$, we have $(r,s) \in \{(m-1, m-3), (m-3, m-1)\}$; by symmetry we may assume $(r,s) = (m-1, m-3)$. (Note also that m cannot be 2, so A is not Boolean.) We claim that m has a divisor m' such that $\ell \ge 2m'-3$. If m is not prime, then it has a divisor m' with $1 < m' < m$. Moreover, $2m'-3 \le m-3 \le \ell$. If m is prime, then condition (ii) holds, so $\ell \neq m-3$. Since $\ell \equiv m-3 \pmod{m}$, it follows that $\ell \ge 2m-3$, so m itself is the sought divisor. It is well-known that in any finite Abelian group, for any divisor of the characteristic, there is an element of that order. Hence there is an element $a \in A$ of order m'.

Let $q = n/m'$ and let $S_1, \ldots, S_q$ be the cosets of $\langle a \rangle$ in A, with $S_q = \langle a \rangle$. Since $r = m-1$, we may write $k = cm'-1$ for some c; now $|N(u)| = cm'$. Fix an element $b \in A \setminus \{a\}$ with order at least 3; this is possible since, as noted above, A is not Boolean. Since $b \notin \langle a \rangle$, we have $-b \notin S_1$. Now $\ell \ge 2m'-3$ implies $c \le q-2$, so by symmetry we may assume $-b \in S_{c+1}$.

We are now ready to construct an A-rainbow labeling of T. Assign label a to vertex u, label b to vertex v, label $-b$ to vertex w, and label 0 to vertex z. To the unlabeled vertices in $N(u)$, arbitrarily assign the $m'-1$ remaining labels in S_1 along with all labels in $S_2, \ldots, S_c$. Now assign all remaining labels to the unlabeled vertices in $N(z)$. Note that all colors in $S_1 \cup \ldots \cup S_c$ appear on edges incident to u and that all colors in $S_{c+1} \cup \ldots \cup S_{q-1}$ appear on edges incident to x. Since z has label 0 and labels $2a, \ldots, (m'-1)a$ all appear on vertices adjacent to z, these same colors appear on edges incident to x. Finally, edge vw has color 0.

Thus every element of A appears on some vertex of T and no element of A appears on more than one edge, so we have an A-rainbow labeling. ■

We close this section by presenting two more classes of caterpillars that fail to have rainbow labelings over groups of characteristic 3.

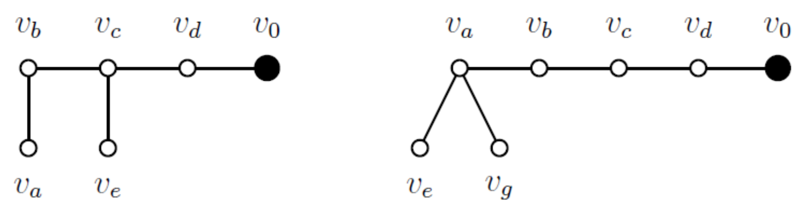

Figure 1: $C[1, 1, 0, \bullet]$ and $C[2, 0, 0, 0, \bullet]$

Theorem 5.7. *1) If A is an Abelian group with characteristic 3, then the caterpillar $C[1, 1, 0, \bullet]$ shown on the left in Figure 1 has no closed labeling over A in which the vertex at $\bullet$ receives label 0.*
2) If A is an Abelian group with characteristic 3, then the caterpillar $C[2, 0, 0, 0, \bullet]$ shown on the right in Figure 1 has no closed labeling over A in which the vertex at $\bullet$ receives label 0.

Proof. 1) Suppose to the contrary that we have such a closed labeling λ with induced edge-coloring λ^*, and let $0, a, b, c, d$, and e be the labels used, with $\lambda(v_0) = 0$, $\lambda(v_a) = a$, $\lambda(v_b) = b$, and so on. View the given caterpillar as a rainbow subtree of K_A, and extend it to a spanning tree T by adding edges from v_0 to all unused vertices. Since λ was a closed labeling with $\lambda(v_0) = 0$, the colors on these new edges did not appear on the original caterpillar, so T is an RST in K_A. The VE-equation now yields $b+2c+d = -f$, where f denotes the free color. We have several cases to consider.

Case 1: $f = c$. In this case 0 is not free and hence occurs as an edge color. By the VE-equation, $b + d = -3c = 0$. Thus the edge colored 0 cannot be incident to v_b or v_d. We conclude that $\lambda^*(v_c v_e) = 0$, hence $c + e = 0$.

Now consider edge $v_a v_b$. Since label 0 appears only on v_0, the color on an edge not incident to v_0 cannot equal the label on one of its endpoints. Thus the color on $v_a v_b$ cannot be a or b, nor can it be c (which is free), 0 (which appears on $v_c v_e$), or d (which appears on $v_d v_0$). The only remaining possibility is that $a + b = \lambda^*(v_a v_b) = e$.

Finally, consider edge $v_b v_c$. This edge cannot have colors 0, d, or e (which appear on $v_c v_e$, $v_d v_0$, and $v_a v_b$ respectively), nor can it have color c (which is free). The edge furthermore cannot have color b, since it is incident to v_b. Hence $b + c = \lambda^*(v_b v_c) = a$. However, we now have $2a = a + (b + c) = (a + b) + c = e + c = 0$, contradicting the assumption that A has characteristic 3.

Case 2: $f \neq c$. Since c is not free, it must appear as an edge color. This color cannot appear on any edge incident to v_c; the only other edges are $v_a v_b$ and $v_d v_0$. Since $v_d v_0$ has color d, we conclude that $a + b = \lambda^*(v_a v_b) = c$.

Now consider edge v_bv_c. This edge cannot have color b, since it is incident to v_b, nor can it have colors c or d, which already appear on other edges. If $b + c = \lambda^*(v_bv_c) = 0$, then $b+(a+b) = b+c = 0$, thus forcing $a-b = a+2b = 0$. Hence $a = b$, which is impossible. If instead $b + c = \lambda^*(v_bv_c) = a$, then $c = a - b$, so $2c = (a - b) + (a + b) = 2a$, which implies $c = a$, another contradiction. We conclude that $b + c = \lambda^*(v_bv_c) = e$.

Finally, consider edge v_cv_e. The only unused colors remaining are 0, a, and b. If $c + e = \lambda^*(v_cv_e) = 0$, then $0 = c + e = c + (b + c) = 2c + b$, so $c = b$, which is impossible. If instead $c+e = \lambda^*(v_cv_e) = a$, then $(a+b)+(b+c) = c+e = a$. Cancelling a yields $2b+c = 0$, which implies $b = c$, another contradiction. Finally, suppose $c+e = \lambda^*(v_cv_e) = b$. Now $c + (b + c) = c + e = b$, hence $2c = 0$, which is again impossible.

2) Suppose to the contrary that we have such a closed labeling λ with induced edge-coloring λ', and let $0, a, b, c, d, e,$ and g be the labels used, with $\lambda(v_0) = 0$, $\lambda(v_a) = a$, $\lambda(v_b) = b$, and so on.

As in Lemma 5.7, view the given caterpillar as a rainbow subtree of K_A, and extend it to a spanning tree T by adding edges from v_0 to all unused vertices. Since λ is a closed labeling with $\lambda(v_0) = 0$, the labels on these new edges did not appear on the original caterpillar, so T is a rainbow spanning tree in K_A.

Consider the edge v_0v_b, which does not belong to T. This edge has color b; if b is the free color in T or if $b = c + d$, then we can obtain a new rainbow spanning tree T^* from T by adding v_0v_b and removing v_cv_d. In T^*, vertices v_a and v_b have degree 3, and all other vertices aside from v_0 have degree 1. Since the sum of the degrees in T^* is $2\,|A|-2$ and since $|A| \equiv 0 \pmod 3$, it follows that $\deg_{T^*}(v_0) \equiv 1 \pmod 3$. Hence Theorem 2.1 implies that T^* is not A-rainbow, a contradiction.

We conclude that b cannot be the free color, nor can it appear on v_cv_d. Since b is not free, it appears as the color of some edge in the caterpillar. It cannot appear on any edge incident to v_b; the only remaining options are v_av_e and v_av_g. By symmetry we may assume $a + e = \lambda'(v_av_e) = b$.

Next consider edge v_av_b. The color on edge v_av_b cannot be a or b, since the edge is incident to v_a and v_b. Likewise, $\lambda'(v_av_b) \neq d$, since color d already appears on v_dv_0. If $a+b = \lambda'(v_av_b) = 0$, then $0 = a+b = a+(a+e) = -a+e$, which is impossible since $a \neq e$. Likewise, if $a + b = \lambda'(v_av_b) = e$, then $e = a + b = a + (a + e) = -a + e$, so $a = 0$, again impossible. The only remaining options are $\lambda'(v_av_b) = c$ and $\lambda'(v_av_b) = g$. We thus have the following facts:

$$\lambda'(v_av_e) = b; \quad \lambda'(v_dv_0) = d; \quad \lambda'(v_av_b) \text{ is either } c \text{ or } g.$$

Case 1: $a + b = \lambda'(v_av_b) = c$. In this case, $\lambda'(v_bv_c) = b + c = b + (a + b) = a - b = a-(a+e) = e$. Now $e-a+b+c = (e+a)+(a+b)+c = (b+c)+c = e+c$, hence $a = b$, a contradiction.

Case 2: $a + b = \lambda'(v_av_b) = g$. In this case, $\lambda'(v_av_g) = a + g = a + (a + b) = -a + b = e$. Only the colors on v_bv_c and v_cv_d remain undetermined; neither can be c, and the only remaining options are 0 and a.

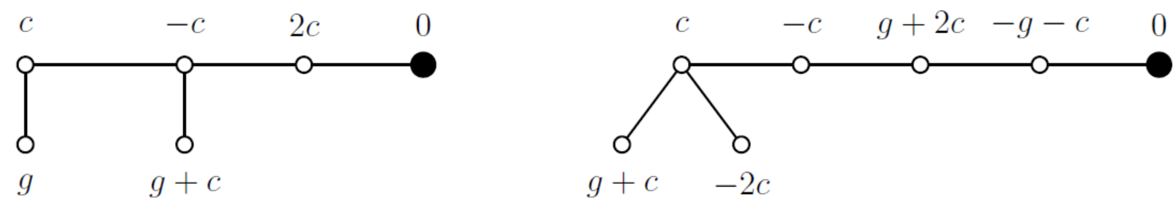

Figure 2: Closed labelings of $C[1, 1, 0, \bullet]$ and $C[2, 0, 0, 0, \bullet]$.

If $b + c = 0$ and $c + d = a$, then $g = \lambda'(v_a v_b) = a + b = c + d + b = d$, a contradiction. If instead $b + c = a$ and $c + d = 0$, then $a + (c + d) = a = b + c = (a + e) + c$. Canceling $a + c$ from both ends yields $d = e$, another contradiction.

■

As a direct consequence of Theorem 5.7, we obtain the following:

Corollary 5.8. *For $k \geq 2$, neither $C[1, 1, 0, 3^k - 6]$ nor $C[2, 0, 0, 0, 3^k - 7]$ has a rainbow labeling over $(\mathbb{Z}_3)^k$.*

Proof. Let $T = C[1, 1, 0, 3^k - 6]$ and let x be the vertex of the spine having $3^k - 6$ incident hairs. If T is $(\mathbb{Z}_3)^k$-rainbow, then some $(\mathbb{Z}_3)^k$-rainbow labeling places label 0 on vertex x. (Given any other rainbow labeling λ, subtracting $\lambda(x)$ from all vertices yields a rainbow labeling with the this property.) Since all labels on neighbors of x appear also as colors on incident hairs, λ restricts to a closed labeling of $C[1, 1, 0, \bullet]$ in which x has label 0; by Theorem 5.7, this is impossible. The non-existence of rainbow labelings for $C[2, 0, 0, 0, 3^k - 7]$ follows similarly from Theorem 5.7. ■

For reference, we note that both caterpillars shown in Figure 1 have closed labelings of the desired form over all other non-Boolean Abelian groups (with at least 6 and 7 elements, respectively). See Figure 2, in which c denotes an element of maximum order in A, and g denotes $3c$ if A is cyclic, and any element not in $\langle c \rangle$ otherwise. (Note that c necessarily has order at least 4.) Such labelings extend naturally to rainbow labelings.

6 Computational Results

We conclude with an exhaustive list of trees that fail to be rainbow over Abelian groups with order at most 20. These lists were determined computationally, by checking all trees on at most 20 vertices for rainbow labelings over all Abelian groups of the same order. Each tree was checked by testing all possible permutations of labels on the tree's vertices (up to certain symmetries); for more details, please refer to the source code at `http://math.uri.edu/~billk/research/rainbow_trees.cpp`. The authors are indebted to Brendan McKay [16] for providing comprehensive catalogues of small trees.

All trees considered were found to have rainbow labelings over the appropriate cyclic groups. Furthermore, we provide no data on rainbow labelings over Boolean groups, as this topic was studied extensively in [11, 12, 20]. Up to isomorphism, there are 8 non-Boolean non-cyclic Abelian groups with order at most 20. To facilitate comparisons, we organize these groups according to their order in Table 1.

All but three of the non-rainbow trees given in Table 1 are caterpillars. The non-caterpillars are shown in Figure 3, where a number followed by a $\times$ at a vertex indicates the number of pendant leaves attached to that vertex. Both Υ_2 and Υ_3 are non-rainbow over $\mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_2$. However, it seems difficult to give simple justifications for this, so we must rely on computational evidence. As Theorems 3.2 and 3.3 indicate, it is difficult to infer much about the global structure of rainbow labelings from just the local structure. That is, if we label only a few vertices of a tree, then we typically cannot determine whether or not this partial labeling can be extended to a rainbow labeling of the full tree. Hence in general, we cannot rule out a partial labeling until it includes a large portion of the tree. This keeps us from narrowing down the enormous number of labelings over A to a manageable size. It seems likely any simple justification for the lack of rainbow labelings of Υ_2 and/or Υ_3 over $\mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_2$ would suggest additional infinite families of non-rainbow trees.

In all other cases, the lack of a rainbow labeling has been justified earlier in the paper.

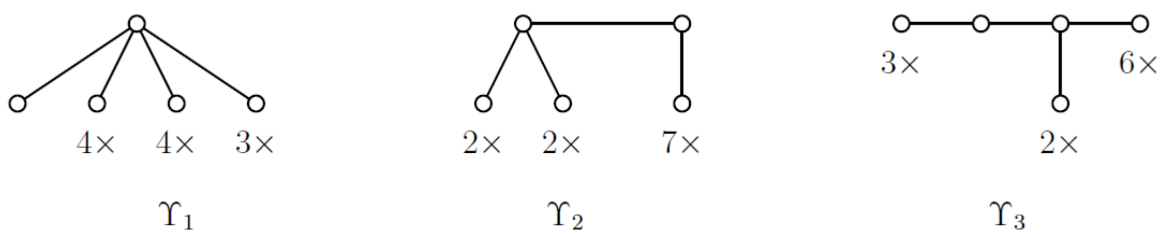

Figure 3: Non-caterpillars having no rainbow labelings

			Justification			
Order	Char.	Group	Theorem 2.1	Thms. 5.4,5.6	Cor. 5.8	Unknown
8	4	$\mathbb{Z}_4 \times \mathbb{Z}_2$	$C[3,3]$	$C[2,0,3]$ $C[2,0,0,2]$	–	–
9	3	$\mathbb{Z}_3 \times \mathbb{Z}_3$	$C[2,5]$ $C[2,1,3]$	$C[1,0,5]$ $C[2,0,4]$ $C[1,0,0,4]$	$C[1,1,0,3]$ $C[2,0,0,0,2]$	–
12	6	$\mathbb{Z}_3 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	$C[5,6]$	$C[4,0,5]$ $C[4,0,0,4]$	–	–
16	4	$\mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	$C[3,11]$ $C[7,7]$ $C[3,2,8]$ $C[3,6,4]$ $C[7,2,4]$ $C[3,2,3,4]$ $C[4,2,2,4]$ Υ_1	$C[2,0,11]$ $C[6,0,7]$ $C[10,0,3]$ $C[2,0,0,10]$ $C[6,0,0,6]$	–	Υ_2 Υ_3
16	4	$\mathbb{Z}_4 \times \mathbb{Z}_4$	$C[3,11]$ $C[7,7]$ $C[3,2,8]$ $C[3,6,4]$ $C[7,2,4]$ $C[3,2,3,4]$ $C[4,2,2,4]$ Υ_1	$C[2,0,11]$ $C[6,0,7]$ $C[10,0,3]$ $C[2,0,0,10]$ $C[6,0,0,6]$	–	–
16	8	$\mathbb{Z}_8 \times \mathbb{Z}_2$	$C[7,7]$	$C[6,0,7]$ $C[6,0,0,6]$	–	–
18	6	$\mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_2$	$C[5,11]$ $C[5,4,6]$	$C[4,0,11]$ $C[10,0,5]$ $C[10,0,0,4]$	–	–
20	10	$\mathbb{Z}_5 \times \mathbb{Z}_2 \times \mathbb{Z}_2$	$C[9,9]$	$C[8,0,9]$ $C[8,0,0,8]$	–	–

Table 1: Non-rainbow trees of order at most 20

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