Section 4.3

Computation of determinants and Cramer's Rule

Introduction

- To compute the determinant of a n x n matrix using the cofactor expansion requires roughly n! operations. (n! = 1 * 2 * 3 . . . * n).

- Consider a 25 x 25 matrix. This would require 25! operations or roughly $1.5 \times 10^{25}$ operations.

- Suppose you have a super computer that can do 1 trillion operations per second. This calculation would require 500,000 years!!

How do you find the determinant of 25 x 25 matrix?

Computation of a Determinant

********************************************************************

Computation of a determinant of a n x n matrix A:

1.) Reduce A to an echelon form, using only row additions and row interchanges.

2.) If any of the matrices appearing in the reduction contains a row of zeros, then det(A) = 0.

3.) Otherwise,

\[ \text{det}(A) = (-1)^r \text{ (Product of pivots)} \]

where \( r \) is the number of row interchanges performed.

**********************************************************************

ROW OPERATIONS

**************************

Property 2: If two different rows of a square matrix A are interchanged,
the determinant of the resulting matrix is $-\det(A)$.

***********************************************************************

***********************************************************************

**Property 4**: If a single row of a square matrix $A$ is multiplied by a scalar $r$, the determinant of the resulting matrix is $r \cdot \det(A)$.

***********************************************************************

***********************************************************************

**Property 5**: If the product of one row of a square matrix $A$ by a scalar is added to a different row of $A$, the determinant of the resulting matrix is the same as the $\det(A)$.

***********************************************************************

Example 1:

```plaintext
> A := matrix([[2,5,7],[6,4,2],[8,4,1]]);

A :=
\[
\begin{bmatrix}
2 & 5 & 7 \\
6 & 4 & 2 \\
8 & 4 & 1 \\
\end{bmatrix}
\]

> A := addrow(A,1,2,-A[2,1]/A[1,1]);

A :=
\[
\begin{bmatrix}
2 & 5 & 7 \\
0 & -11 & -19 \\
8 & 4 & 1 \\
\end{bmatrix}
\]

> A := addrow(A,1,3,-A[3,1]/A[1,1]);

A :=
\[
\begin{bmatrix}
2 & 5 & 7 \\
0 & -11 & -19 \\
0 & -16 & -27 \\
\end{bmatrix}
\]

> A := addrow(A,2,3,-A[3,2]/A[2,2]);
```
\[
A = \begin{bmatrix}
2 & 5 & 7 \\
0 & -11 & -19 \\
0 & 0 & \frac{7}{11} \\
\end{bmatrix}
\]

\[ \text{det}(A) = A[1,1]*A[2,2]*A[3,3]; \]

\[ \text{det}(A) = -14 \]

Example 2:

\[ A := \text{matrix}([[1,-3,1,-2],[2,-5,-1,-2],[0,-4,5,1],[-3,10,-6,5]]); \]

\[
A = \begin{bmatrix}
1 & -3 & 1 & -2 \\
2 & -5 & -1 & -2 \\
0 & -4 & 5 & 1 \\
-3 & 10 & -6 & 5 \\
\end{bmatrix}
\]

Row reducing and then computing the determinant requires roughly \( \frac{2}{3} n^3 \) operations.

Consider a 25 x 25 matrix. This would require \( \frac{2}{3} 25^3 = 10,500 \) operations. Less than 1 second to compute.

Cramer's Rule

Cramer's Rule for solving Systems of Linear Equations

Cramer's rule is a method, based on determinants, for solving a system of linear equations.

The system must be a square system and the coefficient matrix must be nonsingular;
that is; its determinant is nonzero.

**Example 3:** Consider the system

```math
\begin{align*}
  x - 3y + 4z &= 2 \\
  -x - 4y + 3z &= -2 \\
  2x - 5y + 6z &= 5
\end{align*}
```

The coefficient matrix is:

```math
C := \text{matrix}([[1,-3,4],[-1,-4,3],[2,-5,6]]);
```

```
C :=
\begin{bmatrix}
  1 & -3 & 4 \\
  -1 & -4 & 3 \\
  2 & -5 & 6
\end{bmatrix}
```

Right hand side is,

```math
b := \text{matrix}([[2],[-2],[5]]);
```

```
b :=
\begin{bmatrix}
  2 \\
  -2 \\
  5
\end{bmatrix}
```

```math
\text{evalm}(C) \times \text{matrix}(3,1,[x,y,z]) = \text{evalm}(b);
```

```
\begin{bmatrix}
  1 & -3 & 4 \\
  -1 & -4 & 3 \\
  2 & -5 & 6
\end{bmatrix}\begin{bmatrix}
  x \\
  y \\
  z
\end{bmatrix}
= \begin{bmatrix}
  2 \\
  -2 \\
  5
\end{bmatrix}
```
Construct the matrix $A_1$ obtained from matrix $C$ by replacing the first column of $C$ with the right side of the system:

$$A_1 := \begin{bmatrix} 2 & -3 & 4 \\ -2 & -4 & 3 \\ 5 & -5 & 6 \end{bmatrix}$$

Find the value of $x$

$$x = \frac{\det(A_1)}{\det(C)} = 3$$

Construct the matrix $A_2$ obtained from matrix $C$ by replacing the second column of $C$ with the right side of the system:

$$A_2 := \begin{bmatrix} 1 & 2 & 4 \\ -1 & -2 & 3 \\ 2 & 5 & 6 \end{bmatrix}$$

Find the value of $y$

$$y = \frac{\det(A_2)}{\det(C)} = -1$$

Construct the matrix $A_3$ obtained from matrix $C$ by replacing the third column of $C$ with the right side of the system:

$$A_3 := \begin{bmatrix} 1 & -3 & 2 \\ -1 & -4 & -2 \\ 2 & -5 & 5 \end{bmatrix}$$
Find the value of $z$

$$z = \frac{\det(A^3)}{\det(C)};$$

$$z = -1$$

Note that the solution coincides with the solution obtained using

$$\text{solve}\{x-3*y+4*z=2,-x-4*y+3*z=-2,2*x-5*y+6*z=5\},\{x,y,z\};$$

$$\{y = -1, z = -1, x = 3\}$$

>  

**Adjoint matrix**

***********************************************************************

The steps for finding the adjoint matrix of $A$.

1. Find the cofactor of each entry of the matrix $A$. The cofactor of an entry $a_{ij}$ is defined as:

$$C_{ij} = (-1)^{i+j} \cdot \det(M_{ij})$$

where the submatrix $M_{ij}$ is the minor of the entry $a_{ij}$.

2. Replace each entry of matrix $A$ by its cofactor to get a new matrix $C$. This matrix is called the **cofactor matrix**.

3. The transpose of matrix $C$ is called the **adjoint matrix** of $A$ and is denoted by $\text{Adj}(A)$.

***********************************************************************

**Example 4:** Find the adjoint of the matrix

$$A := \text{matrix}([[1,3,5],[5,3,6],[8,4,2]]);$$

$$A = \begin{bmatrix} 1 & 3 & 5 \\ 5 & 3 & 6 \\ 8 & 4 & 2 \end{bmatrix}$$

The minors of all of the entries are respectively given by
\[ C_{11} := (-1)^2 \text{det}(M_{11}); \]
\[ C_{12} := (-1)^3 \text{det}(M_{12}); \]
\[ C_{13} := (-1)^4 \text{det}(M_{13}); \]

\[ C_{21} := (-1)^3 \text{det}(M_{21}); \]
\[ C_{22} := (-1)^4 \text{det}(M_{22}); \]
\[ C_{23} := (-1)^5 \text{det}(M_{23}); \]

\[ C_{31} := (-1)^4 \text{det}(M_{31}); \]
\[ C_{32} := (-1)^5 \text{det}(M_{32}); \]
\[ C_{33} := (-1)^6 \text{det}(M_{33}); \]

The cofactor matrix is,

\[
\begin{pmatrix}
C_{11} & C_{12} & C_{13} \\
C_{21} & C_{22} & C_{23} \\
C_{31} & C_{32} & C_{33}
\end{pmatrix}
\]

\[
\begin{bmatrix}
-18 & 38 & -4 \\
14 & -38 & 20 \\
3 & 19 & -12
\end{bmatrix}
\]

The adjoint of A is

\[
\text{\textasciitilde Adj}(A) = \text{transpose}(C);
\]
Result:

Let \( A = \begin{bmatrix} a_{ij} \end{bmatrix} \) be an \( n \times n \) matrix. If \( C_{ij} = (-1)^{i+j} \) *\( \det( M_{ij} ) \) denotes the cofactor for \( a_{ij} \) then

\[
\begin{align*}
    a_{i1} C_{k1} + a_{i2} C_{k2} + a_{i3} C_{k3} + \cdots + a_{in} C_{kn} &= \det(A) \text{ if } k = i \\
    &= 0 \text{ if } k \neq i
\end{align*}
\]

Example 5:

\[
A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}
\]

The minors of entries \( a_{11}, a_{12}, a_{13} \) are respectively given by

\[
M11 := \text{minor}(A,1,1); \\
M12 := \text{minor}(A,1,2); \\
M13 := \text{minor}(A,1,3);
\]
The cofactor $C_{ij}$ of the entry $a_{ij}$ is $(-1)^{i+j} \cdot \det(M_{ij})$;

$$C_{11} := (-1)^{2} \cdot \det(M_{11})$$
$$C_{12} := (-1)^{3} \cdot \det(M_{12})$$
$$C_{13} := (-1)^{4} \cdot \det(M_{13})$$

The determinant of $A$.

$$\text{det}(A) = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$$

Value should be zero.

$$\text{simplify}(a_{21}C_{11} + a_{22}C_{12} + a_{23}C_{13})$$

0

Let $A$ be an $n \times n$ nonsingular matrix. Then the inverse of $A$ is given by:

$$A^{-1} = \frac{\text{Adj}(A)}{\det(A)}$$

Example 6 Find the inverse of the following matrix
\[ A := \begin{bmatrix} 1 & 3 & 5 \\ 5 & 3 & 6 \\ 8 & 4 & 2 \end{bmatrix} \]

\[ \text{Adjoint}(A) := \begin{bmatrix} -18 & 14 & 3 \\ 38 & -38 & 19 \\ -4 & 20 & -12 \end{bmatrix} \]

The product of \( A \) with its adjoint is the matrix

\[ \text{multiply}(A, \text{Adjoint}(A)) \]

This is the identity matrix multiplied by the determinant of the matrix \( A \).

\[ \text{det}(A) \]

\[ A^{-1} = \frac{\text{Adjoint}(A)}{\text{det}(A)} \]

**Exercises**

1, 3, 5, 7, 9, 15, 17, 19, 21, 22, 25, 27, 29.