Chapter 3
3.10

Suppose that \( n \in \mathbb{N} \) and that \( x_1, \ldots, x_{2n+1} \) are odd integers. Prove that \( \prod_{i=1}^{2n+1} x_i \) is odd.

**Proof:** (by induction)

**Base Case:**

\( P(0) \): There is one odd number in the product.

Therefore, the product is odd.

We assume \( P(k) \).

\( P(k) \): The product of any \( 2k+1 \) odd numbers is odd.

We prove \( P(k+1) \).

\( P(k+1) \): The product of any \( 2(k+1)+1 = 2k+3 \) odd numbers is odd.

Let \( x_1, \ldots, x_{2k+3} \) be odd numbers.

\[ \Pi_{i=1}^{2k+3} x_i = \Pi_{i=1}^{2k+1} x_i x_{2k+2} x_{2k+3} \]

We claim that the product of three odd numbers is odd.

Let \( an_1 + 1, an_2 + 1, an_3 + 1 \) be odd.

\[ (an_1 + 1)(an_2 + 1)(an_3 + 1) = 4n_1n_2 + 2n_1 + 4n_1n_3 + 2n_1 + 2n_2 + 4n_2n_3 + 2n_2 + 2n_3 + 1 = \]

\[ 2(2n_1n_2 + n_1 + n_2 + n_3 + n_1n_3) + 3 = 2(2n_1n_2 + n_1 + n_2 + n_3 + n_1n_3) + 3 \]

which is odd.

So \( \Pi_{i=1}^{2k+3} x_i \) is the product of three odd numbers, which is odd by the claim. \( \blacksquare \)
For \( n \in \mathbb{N} \), prove that \( \sum_{i=1}^{n} i^3 = \left( \frac{n(n+1)}{2} \right)^2 \).

Using induction to establish the basis case \( P(1) \), then by assuming \( P(k) \) to be true, we establish \( P(k+1) \) is true.

1. **Basis Step** \( P(1) \):
   \[
   \sum_{i=1}^{1} i^3 = \left( \frac{1(1+1)}{2} \right)^2
   \]
   \[
   \text{LHS} = \sum_{i=1}^{1} i^3 = (1)^3 = 1
   \]
   \[
   \text{RHS} = \left( \frac{1(1+1)}{2} \right)^2 = \left( \frac{3}{2} \right)^2 = 1
   \]
   \[
   \text{LHS} = \text{RHS} = 1 \checkmark
   \]

2. **Assume \( P(k) \) is True**
   \( P(k) \):
   \[
   \sum_{i=1}^{k} i^3 = \left( \frac{k(k+1)}{2} \right)^2
   \]

3. **Prove \( P(k+1) \):**
   \[
   \sum_{i=1}^{k+1} i^3 = \left( \frac{(k+1)(k+2)}{2} \right)^2
   \]
   \[
   \text{LHS} = \sum_{i=1}^{k+1} i^3 = \sum_{i=1}^{k} i^3 + (k+1)^3
   \]
   \[
   = \left( \frac{k(k+1)}{2} \right)^2 + (k+1)^3
   \]
   \[
   = \frac{k^2(k+1)^2}{4} + \frac{4(k+1)^3}{4}
   \]
   \[
   = \frac{k^2(k+1)^2 + 4(k+1)^3}{4}
   \]
   \[
   = \frac{(k+1)^2(k^2+4k+4)}{4}
   \]
   \[
   = \frac{(k+1)^2(k+2)^2}{4}
   \]
   \[
   \text{LHS} = \sum_{i=1}^{k+1} i^3 = \left( \frac{(k+1)(k+2)}{2} \right)^2
   \]
   \[
   \text{RHS} = \left( \frac{(k+1)(k+2)}{2} \right)^2
   \]
   \[
   \text{LHS} = \text{RHS} = \left( \frac{(k+1)(k+2)}{2} \right)^2 \checkmark
3.16
For \( n \in \mathbb{N} \), (where \( \mathbb{N} \) is the set of Natural Numbers \( \{1,2,3,\ldots\} \)) prove that

\[
\sum_{i=1}^{n} i^3 = \frac{n(n+1)^2}{2}
\]

**Proof by Induction:**

**Base Case:** \( P(1) \):
Left-Hand-Side (LHS) = \( (1)^3 = 1 \)
Right-Hand-Side (RHS) = \( \left( \frac{1(1+1)}{2} \right)^2 = 1 \)

so \( \text{RHS} = \text{LHS} \). ✓

**Assume** \( P(k) \) is true:

\( P(k): \sum_{i=1}^{k} i^3 = \left( \frac{k(k+1)}{2} \right)^2 \)

**Prove** \( P(k+1) \):

\[
\sum_{i=1}^{k+1} i^3 = \left( \frac{(k+1)(k+2)}{2} \right)^2
\]

\[
\text{LHS} = \sum_{i=1}^{k+1} i^3 = \sum_{i=1}^{k} i^3 + (k+1)^3
\]

\[
= \left( \frac{k(k+1)}{2} \right)^2 + (k+1)^3
\]

\[
= \frac{k^2(k+1)^2 + 4(k+1)^3}{4}
\]

\[
= (k+1)^2 \left[ \frac{k^2 + 4(k+1)}{4} \right]
\]

\[
= (k+1)^2 \left( \frac{(k+1)(k+2)}{2} \right)^2
\]

\[
\text{RHS} = \left( \frac{(k+1)(k+2)}{2} \right)^2
\]

so \( \text{LHS} = \text{RHS} \).

✓

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3.16 VIkiya Chea

For \( n \in \mathbb{N} \), prove that \( \sum_{i=1}^{n} i^3 = \left( \frac{n(n+1)}{2} \right)^2 \)

**Proof:** The base case is for \( n = 1 \)

\[
\sum_{i=1}^{1} i^3 = \left( \frac{1(1+1)}{2} \right)^2
\]

The left hand side gives 1 as well as the right hand side. Because the left hand side equals the right hand side, \( P(1) \) is true.

The induction assumption:

Assume \( P(k) \) is true that is

\[
\sum_{i=1}^{k} i^3 = \left( \frac{k(k+1)}{2} \right)^2
\]

Now prove the result for \( n = k+1 \), that is

\[
\sum_{i=1}^{k+1} i^3 = \left( \frac{(k+1)(k+2)}{2} \right)^2
\]

\[
= \left( k+1 \right)^2 \left[ \frac{(k+1)^3 + 2(k+1) + 1}{4} \right]
\]

\[
= \left( k+1 \right)^2 \left[ \frac{k^2 + 4k + 4}{4} \right]
\]

This is equal to the right hand side.

The left hand side is as follows:

\[
\sum_{i=1}^{k+1} i^3 = \sum_{i=1}^{k} i^3 + (k+1)^3
\]

\[
= \left( \frac{k(k+1)^2}{2} + \frac{4(k+1)^3}{4} \right)
\]

\[
= (k+1)^2 \left[ \frac{k^2 + 4k + 4}{4} \right] = (k+1)^2 \left[ \frac{2(k^2 + 4k + 4)}{4} \right]
\]

This is equal to the right hand side.