1.29: Let $x, y, z$ be non-negative real numbers such that $y + z \geq 2$. Prove that $(x + y + z)^2 \geq 4x + 4yz$. Determine when equality holds.

We reassociate $(x + y + z)^2$ to $[x + (y + z)]^2$, and multiply it out to find $(x + y + z)^2 = x^2 + 2x(y + z) + (y + z)^2$. We now consider each term of this sum separately. As $x \in \mathbb{R}$, $x^2 \geq 0$. It is given that $y + z \geq 2$; we multiply both sides of this by $2x$, and noting that as we are given $x$ is non-negative the inequality does not reverse, giving us $2x(y + z) \geq 4x$. By the Arithmetic-Geometric Mean Inequality, $(y + z)^2 \geq 4yz$. By putting these parts together, we arrive at $x^2 + 2x(y + z) + (y + z)^2 \geq 0 + 4x + 4yz$, so $(x + y + z)^2 \geq 4x + 4yz$, q.e.d.

To determine when equality holds, we reconsider the first and third partial inequalities and calculate them as equalities. As $x^2 \geq 0$ becomes $x^2 = 0$, $x = 0$. As $(y + z)^2 \geq 4yz$ is a form of the Arithmetic-Geometric Mean Inequality, $(y + z)^2 = 4yz$ holds only when $y = z$. Therefore, $(x + y + z)^2 = 4x + 4yz$ if and only if $x = 0$ and $y = z$. 

\[\checkmark\]
1.29. Let $x, y, z$ be nonnegative real numbers such that $y + z \geq 2$. Prove that

$$(x + y + z)^2 \geq 4x + 4yz.$$ Determine when equality holds.

**Proof.** We begin with the inequality $y + z \geq 2$. Subtract 2 from each side to give

$$y + z - 2 \geq 0.$$ Multiplying both sides by $2x$ to get $2xy + 2xz - 4x \geq 0$.

We can now add $x^2$ to the left side of the inequality. This does not change the inequality because $x^2 \geq 0$. The new equation is $x^2 + 2xy + 2xz - 4x \geq 0$. We add $(y - z)^2$ to the left side of the inequality. This will always be a positive number regardless of the value of $y$ and $z$ and will not change the inequality because $(y - z)^2 \geq 0$. This yields the equation $(y - z)^2 + x^2 + 2xy + 2xz - 4x \geq 0$.

*Note that equality holds only when $y = z$ and $x = 0$.

We continue the proof by expanding $(y - z)^2$ to get

$$(y^2 - 2yz + z^2) + x^2 + 2xy + 2xz - 4x \geq 0.$$ We can rewrite $-2yz$ as $-4yz + 2yz$ and place it into the equation to give $y^2 - 4yz + z^2 + x^2 + 2xy + 2xz - 4x \geq 0$ or

$$(x + y + z)(x + y + z) - 4x - 4yz \geq 0.$$ Finally, add $4x + 4yz$ to both sides to get

$$(x + y + z)^2 \geq 4x + 4yz.$$