Section 1.2: #8: For any two finite sets \( S \) and \( T \), show that the average of \( |S| \) and \( |T| \) does not exceed \( |S \cup T| \).

The average of \( |S| \) and \( |T| \) is \( \frac{|S| + |T|}{2} \).

Since \( S \cup T \) contains all of the elements in \( S \) and all of the elements in \( T \), \( S \subseteq S \cup T \) and \( T \subseteq S \cup T \). Thus by theorem 1.2.1 part c, \( |S| \leq |S \cup T| \) and \( |T| \leq |S \cup T| \).

So: \( |S| + |T| \leq |S \cup T| + |S \cup T| \).

\[ |S| + |T| \leq 2|S \cup T| \]

\[ |S| + |T| \leq 2|S \cup T| \]

Therefore, the average of \( |S| \) and \( |T| \) does not exceed \( |S \cup T| \).
**Proposition 1(3):**

\[ (-x)y = -(xy) \]

In order to prove Proposition 1(3) we need to show that \((-x)y\) is the additive inverse of \(xy\). To show this we need to prove that \((-x)y + (xy) = 0\). Using the distributive law we have that \((x + (-x))y = 0\). Using the additive inverse we get \(0 \cdot y = 0\). By the communitivity property we can say that \(y \cdot 0 = 0\). Using Proposition 1(2), which states that \(y \cdot 0 = 0\), we have that \(0 = 0\). This tells us that \((-x)y\) is the additive inverse of \(xy\). The additive inverse of \(xy\) is \(-xy\). This proves that \((-x)y = -(xy)\). \(\checkmark\)
Section 1-4 #9 Use a truth table to verify that \((P \land \sim Q) \lor (\sim P \land Q) \lor (\sim P \lor Q)\) is a tautology.

\textbf{Proof:}

\begin{center}
\begin{tabular}{cc|cccc|cccc|cccc}
\hline
P & Q & \sim P & \sim Q & P \land Q & \sim P \land Q & \sim P \lor Q & (P \land \sim Q) \lor (\sim P \land Q) & (P \land \sim Q) \lor (\sim P \land Q) \lor (\sim P \lor Q) \\
\hline
T & T & F & F & F & F & F & F & T \\
T & F & F & T & T & T & T & T & T \\
F & T & F & T & F & F & T & T & T \\
F & F & T & T & F & T & F & T & T \\
\hline
\end{tabular}
\end{center}

Since the statement is always true, so it is a tautology.
Given $2 \leq y + z$, show that $4x + 4yz \leq (x + y + z)^2$. Also assume $x \geq 0$.

Theorem 1 of the AGM Inequality states:

"If $x, y$ are an element of the real numbers, then $\frac{xy}{(x + y)^2} \leq \frac{1}{2}$.

If we let $x = y$ and $y = z$ we get the following:

$$yz \leq \frac{(y + z)^2}{(2)^2}.$$  

$$yz \leq \frac{(y + z)^2}{4}.$$  

$$4yz \leq (y + z)^2.$$  

Also $(x + (y + z))^2 = x^2 + 2x(y + z) + (y + z)^2$. We can add $x^2 + 2x(y + z)$ to both sides of this inequality to get the following:

$$x^2 + 2x(y + z) + 4xy \leq (x + y + z)^2.$$  

Now we are going to use what we are given which is, $2 \leq y + z$. Since $x \geq 0$ we can multiply $2x$ to both sides of this inequality. As a result we get the following inequality:

$$2(2x) \leq 2x(y + z).$$

$$4x \leq 2x(y + z).$$

So now we have $x^2 + 4x + 4xy \leq (x + y + z)^2$. We also can use Proposition 4(4), which states, $0 \leq x^2$ which is true for all $x$. So we can say $4x + 4yz \leq (x + y + z)^2$. □
Section 1.6, #6a.b

For the following statements:

\[ x^2 = 4 \Rightarrow x = 2 \] \hspace{1cm} (4)
\[ \text{If } 2x \leq x, \text{ then } x^2 > 0 \] \hspace{1cm} (5)

Find the converse, and determine the truth value for both the statement and its converse. Justify.

(1)
\[ (P \Rightarrow Q) \quad x^2 = 4 \Rightarrow x = 2 \quad F \]

**Proof:** Let \( x = -2 \Rightarrow x^2 = 4 \Rightarrow x = 2 \). So, a contradiction is found.
\[ \therefore (P \Rightarrow Q) \text{ is } F. \]
\[ (Q \Rightarrow P) \quad x = 2 \Rightarrow x^2 = 4 \quad T \]

**Proof:** Let \( x = 2 \Rightarrow x^2 = 4 \).
\[ \therefore (Q \Rightarrow P) \text{ is } T. \]

(2)
\[ (P \Rightarrow Q) \quad \text{If } 2x \leq x, \text{ then } x^2 > 0 \quad F \]

**Proof:** Let \( x = 0 \). \( 2 \cdot 0 \leq 0, \text{ but } 0^2 = 0 > 0 \) is false. So, a contradiction is found.
\[ \therefore (P \Rightarrow Q) \text{ is } F. \]
\[ (Q \Rightarrow P) \quad \text{If } x^2 > 0, \text{ then } 2x \leq x \quad F \]

**Proof:** Let \( x = 2 \Rightarrow 4 > 0 \Rightarrow 4 \leq 2 \). But \( 4 > 2 \), so a contradiction is found.
\[ \therefore (Q \Rightarrow P) \text{ is } F. \]
Let \( \ell \) be the line \( y = m(x-a) + a^2 \) through the point \((a, a^2)\) with slope \(m\). Prove that \( \ell \) intersects the parabola \( y = x^2 \) in exactly one point if and only if \( m = 2a \).

The proof of this example involves two cases. The first case deals with the statement, if \( \ell \) intersects the parabola \( y = x^2 \) in exactly one point then \( m = 2a \). Assume \( m(x-a) + a^2 = x^2 \)

\[ x^2 - mx + (ma-a^2) = 0. \]

So, \( x = (m \pm \sqrt{m^2 - 4(ma-a^2)})/2 \). And so, \( \sqrt{m^2 - 4(ma-a^2)} = 0 \). Now we have that \( m^2 - 4ma - 4a^2 = 0 \) and \( m = (4a + \sqrt{16a^2 - 4(4a^2)})/2 = 2a \). Therefore if \( \ell \) intersects \( y = x^2 \) in one place if \( m = 2a \). The second case deals with the statement, if \( m = 2a \) then \( \ell \) intersects the parabola \( y = x^2 \) in exactly one point. The line and the parabola intersect when the two equations are equal. In other words the two equation intersect where \( x^2 = 2a(x-a) + a^2 \). Factoring we get that \( x^2 - 2ax - 2a^2 + a^2 = 0 \). Combining like terms you get \( x^2 - 2ax + a^2 = 0 \). Bringing all the variables to one side you get that \( x^2 - 2ax + a^2 = 0 \). In other words \((x-a)^2 = 0 \) or \( x = a \). Therefore if \( m = 2a \), then the line intersects the parabola in exactly one point.
Section 2.2 #2b

Prove $1^3 + 2^3 + 3^3 + \ldots + n^3 = \frac{n^2(n + 1)^2}{4} \quad \forall n \in \mathbb{N}$

Let $n = 1$ \quad LHS = $\sum_{i=0}^{1} i^3 = 0^3 + 1^3 = 1$ \quad RHS = $\frac{1^2(1 + 1)^2}{4} = 1$

Assume true for all $k \geq 1$ \quad $\sum_{i=0}^{k} i^3 = \frac{k^2(k + 1)^2}{4}$

Show for $k + 1$

$$\sum_{i=0}^{k+1} i^3 = \sum_{i=0}^{k} i^3 + (k + 1)^3 = \frac{k^2(k + 1)^2}{4} + (k + 1)(k + 1)^2$$

$$= \frac{k^2(k + 1)^2 + 4k + 4(k + 1)^2}{4}$$

$$= \frac{(k^2 + 4k + 4)(k + 1)^2}{4}$$

$$= \frac{(k + 2)^2(k + 1)^2}{4}$$
Section 2.2. #4

Show that the sum of the cubes of any three consecutive natural numbers is a multiple of 9.

**Proof** \( n^3 + (n + 1)^3 + (n + 2)^3 = 9m \) where \( n, m \in \mathbb{N} \).

Let \( n = 1 \), then \( 1^3 + 2^3 + 3^3 = 36 \) is divisible by 9 \((m = 4)\).

Assume that \( k^3 + (k + 1)^3 + (k + 2)^3 = 9m \) for \( k \geq 1 \) and \( m \in \mathbb{N} \), then we want to show \( (k + 1)^3 + (k + 2)^3 + (k + 3)^3 = 9j \) for some \( j \in \mathbb{N} \)

\[
(k + 1)^3 + (k + 2)^3 + (k + 3)^3 - k^3 + (k + 1)^3 + (k + 2)^3 + (k + 3)^3 - k^3 \\
= 9m + (k + 3)^3 - k^3 \\
= 9m + k^3 + 9k^2 + 27k + 27 - k^3 \\
= 9m + 9(k^2 + 3k + 1) \\
= 9(m + k^2 + 3k + 1) \\
= 9j \ (j = m + k^2 + 3k + 1) \]
Section 3.1, #20

Prove that \(4 | 13^n - 1 \forall n \in \mathbb{N}\).

Using Mathematical Induction:

Base case: \(n = 1\). Then \(13^1 - 1 = 12, 12 = 4 \times 3\), so the base case checks.

Induction Assumption: for \(k \geq 1, 4 | 13^k - 1\). This means that for some \(m \in \mathbb{Z}, 13^k - 1 = 4m\).

We want to show that \(4 | 13^{k+1} - 1\), i.e. \(13^{k+1} - 1 = 4p\) for some \(p \in \mathbb{Z}\).

We know \(13^{k+1} - 1 = (13^k \cdot 13) - 1\) from the exponent laws.

\[
= [(4m+1) \cdot 13] - 1 \text{ from the induction assumption.}
\]

\[
= (52m + 13) - 1 \text{ after distributing 13.}
\]

\[
= 52m + 12 \text{ after subtraction}
\]

\[
= 4(13m + 3) \text{ after factoring out a 4.}
\]

Since \(m \in \mathbb{Z}, (13m + 3) \in \mathbb{Z}\). Therefore, \(13^{k+1} - 1 = 4p\), where \(p = 13m + 3\), and \(p \in \mathbb{Z}\).

Therefore, \(4 | 13^{k+1} - 1\). ◆
4.3, #6  Observe that the solution of Example 4.3.6. almost always involved
the sum of entries of the eleventh row of Pascal's triangle. Conjecture and prove
a formula for the sum of the entries of the n-th row of Pascal's triangle.

**Note**  By the definition of Pascal's triangle, the zeroth row is \( \{1\} \) and the
first row is \( \{1, 1\} \); for \( n > 1 \) and \( 0 < i < n \), the \( i \)-th element of the \( n \)-th row is
equal to the sum of the \( (i - 1) \)-th and \( i \)-th elements of the \( (n - 1) \)-th row and
the first and last elements are always 1. Hence the first few rows are

\[
\begin{array}{cccc}
1 \\
1 & 1 \\
1 & 2 & 1 \\
1 & 3 & 3 & 1 \\
1 & 4 & 6 & 4 & 1 \\
\end{array}
\]

**Lemma**  For \( n > 1 \) the \( n \)-th row of Pascal's triangle has the form

\[
1 \left( \begin{array}{c} n \\ 1 \end{array} \right) \left( \begin{array}{c} n \\ 2 \end{array} \right) \ldots \left( \begin{array}{c} n \\ n - 1 \end{array} \right) 1.
\]

**Proof**  Let \( S_k \) be the statement that for \( k > 1 \) the \( k \)-th row of Pascal's
triangle has the form

\[
1 \left( \begin{array}{c} k \\ 1 \end{array} \right) \left( \begin{array}{c} k \\ 2 \end{array} \right) \ldots \left( \begin{array}{c} k \\ k - 1 \end{array} \right) 1.
\]

For \( k = 2 \), \( S_k \) - the statement that the \( k \)-th row of Pascal's triangle has the form

\[
1 \left( \begin{array}{c} 2 \\ 1 \end{array} \right) 1 = 1 \ 2 \ 1
\]

is true. For some \( k \geq 1 \), assume \( S_k \) is true. By the definition of Pascal's triangle,
the \( (k + 1) \)-th row is

\[
1 \left[ \left( \binom{n}{k} + \binom{k}{1} \right) \left( \binom{k+1}{1} + \binom{k+1}{2} \right) \ldots \left( \binom{k}{k-1} + \binom{k}{k} \right) \right] 1.
\]

Since \( \binom{n}{k-1} + \binom{n}{k} = \binom{n+1}{k} \), we see that \( S_{k+1} \)

\[
1 \left( \begin{array}{c} k+1 \\ 1 \end{array} \right) \left( \begin{array}{c} k+1 \\ 2 \end{array} \right) \ldots \left( \begin{array}{c} k+1 \\ k \end{array} \right) 1
\]

is true. Therefore, by the Principle of Mathematical Induction, \( S_n \) is true for
all \( n \in \mathbb{N} \).
Conjecture  The sum of the entries in the $n$-th row of Pascal’s triangle is $2^n$.

Proof  By the previous Lemma and the identities $\binom{n}{0} = \binom{n}{n} = 1$, the sum of the entries in the $n$-th row of Pascal’s triangle is
\[
\binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{n-1} + \binom{n}{n} = \sum_{i=0}^{n} \binom{n}{i}.
\]

By the Binomial Theorem which states that $(x + y)^n = \sum_{j=0}^{n} \binom{n}{j} x^j y^{n-j}$, and letting $x = y = 1$, we have
\[
\sum_{i=0}^{n} \binom{n}{i} = (1 + 1)^n = 2^n. \quad \blacksquare
\]