Linear Algebra - MTH 215 - Spring 2002

Section 1.3 - The proof that the transpose of a product is the product of the transposes in reverse order

*Theorem:* If $A$ is $m$ by $n$ and $B$ is $m$ by $n$ then

$$(AB)^T = B^T A^T$$

*Proof:* We will show that for any row $i$ and column $j$, the $i, j$-entry on the LHS is equal to the $i, j$-entry on the RHS.

The $i, j$-entry of the LHS is $(AB)^{i,j}_T$. It is the same as the $j, i$-entry of $AB$. That is,

$$(AB)^T_{i,j} = (AB)_{j,i}$$

The $j, i$-entry of $AB$ is (row $j$ of $A$) dot (column $i$ of $B$).

On the other hand, the $i, j$-entry of the RHS is the $i, j$-entry of the product $B^T A^T$. This can be expressed as (row $i$ of $B^T$) dot (column $j$ of $A^T$). Which is the same as saying (column $i$ of $B$) dot (row $j$ of $A$).

So we see that we expressed the $i, j$-entries of each side as the dot product of the same two vectors. Therefore, the LHS = RHS. ☃️