Linear Algebra - MTH 215 - Spring 2002

Section 1.6 - Homogeneous Systems, Subspaces, and Bases

Theory

- Solutions to homogeneous systems
- Span of a set of vectors, subspace.
- Column space of a matrix. $A\mathbf{x} = \mathbf{b}$, $\mathbf{b}$ in the column space of $A$.
- Row space of a matrix.
- Basis, A set of vectors $v_1, v_2, \ldots, v_k$ is a basis for a vector space $W$ if $v_1, v_2, \ldots, v_k$ spans $W$ and is linearly independent. If are looking at a set of vectors $v_1, v_2, \ldots, v_k$ and want to know if they form a basis of the subspace $sp(v_1, v_2, \ldots, v_k)$ then you really just need to know if they are linearly independent.

**Theorem 1** (1.16) The following are equivalent for an $n$ by $n$ matrix $A$.

1. There is a unique solution to $A\mathbf{x} = \mathbf{b}$.
2. $A$ is row equivalent to $I$.
3. $A$ is invertible.
4. The columns of $A$ for a basis for $\mathbb{R}^n$. Every vector $\mathbf{b}$ is in the column space of $A$.

Example

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$
Theorem 2 (1.17) The following are equivalent for an \( m \) by \( n \) matrix \( A \).

1. Each consistent system \( Ax = b \) has a unique solution.

2. \( A \) is row equivalent to \( I \) with rows of zeros at the bottom.

3. The columns of \( A \) for a basis for the column space of \( A \). In other words, the columns are linearly independent.

Example

\[
A = \begin{bmatrix}
1 & 1 & 0 & 0 \\
0 & 1 & 1 & 0 \\
0 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0
\end{bmatrix}
\]

Corollary 3 If \( m < n \) and \( Ax = b \) is consistent there are an infinite number of solutions.

Example

\[
A = \begin{bmatrix}
1 & 1 & 0 & 0 \\
0 & 1 & 1 & 0 \\
0 & 0 & 1 & 1
\end{bmatrix}
\]

Corollary 4

1. If \( Ax = 0 \), \( m < n \) has a nontrivial solution.

2. If \( Ax = 0 \), \( n \) by \( n \) has a nontrivial solution if and only if \( A \) is NOT row equivalent to \( I \).

Example

\[
A = \begin{bmatrix}
1 & 1 & 0 & 0 \\
0 & 1 & 1 & 0 \\
0 & 0 & 1 & 1
\end{bmatrix}
\begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4
\end{bmatrix} = \begin{bmatrix}
0 \\
0 \\
0 \\
0
\end{bmatrix}
\]
Example

\[ A = \begin{bmatrix}
1 & 1 & 0 & 0 \\
0 & 1 & 1 & 0 \\
0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 \\
\end{bmatrix} \begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
\end{bmatrix} = \begin{bmatrix}
0 \\
0 \\
0 \\
0 \\
\end{bmatrix} \]

**Theorem 5** (1.18) If \( p \) is a solution of \( Ax = b \) and \( h \) is a solution of \( Ax = 0 \) then \( p + h \) is a solution of \( Ax = b \). In fact, every solution has this form.

Example

\[ A = \begin{bmatrix}
1 & 1 & 0 & 0 \\
0 & 1 & 1 & 0 \\
0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 \\
\end{bmatrix} \begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
\end{bmatrix} = \begin{bmatrix}
2 \\
-1 \\
1 \\
\end{bmatrix} \]

Example

\[ A = \begin{bmatrix}
1 & 1 & 0 & 0 \\
0 & 1 & 1 & 0 \\
0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 \\
\end{bmatrix} \begin{bmatrix}
x_1 \\
x_2 \\
x_3 \\
x_4 \\
\end{bmatrix} = \begin{bmatrix}
0 \\
0 \\
0 \\
0 \\
\end{bmatrix} \]