Integrating Rational Functions

Step by step.

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Section 7.4 covers integrating rational functions, that is, functions that look like a polynomial divided by a polynomial. You already could handle certain rational functions like:

\[ \int \frac{3x^2 + 7x - 4}{x^3} \, dx \]

by first rewriting \( \frac{3x^2 + 7x - 4}{x^3} \) as \( \frac{3x^2}{x^3} + \frac{7x}{x^3} - \frac{4}{x^3} \) and then integrating each term in the sum separately.

The problem is if the denominator is a polynomial with more than one nonzero coefficient, like:

\[ \int \frac{3x^2 + 7x - 4}{x^3 + 2x^2 - 7x} \, dx. \]

For this we need some new techniques.

The 4 step process reduces rational functions into the sum of rational functions which will be one of the following forms, which you should know how to integrate by previous techniques.

1. \( \int \frac{1}{ax + b} \, dx = \frac{1}{a} \ln |ax + b| + C \)
2. for some \( n > 1 \), \( \int \frac{1}{(ax + b)^n} \, dx = \frac{1}{a(-n+1)} (ax + b)^{-n+1} + C \)
3. \( \int \frac{Ax + B}{ax^2 + bx + c} \, dx \) where \( ax^2 + bx + c \) is irreducible. First complete the square and then use formula 24 or 25 from the table on the inside back cover of our book.

Examples of each of the above are:

1. \( \int \frac{1}{x-2} \, dx = \ln |x - 2| + C \)
2. \( \int \frac{2}{(x-1)^3} \, dx = -\frac{1}{(x-1)^2} + C \)
3. \( \int \frac{x+2}{x+1} \, dx = \frac{1}{2} \ln |x^2 + 1| + 2 \tan^{-1} x + C \)

One further type is more complicated: \( \int \frac{3x-1}{(x^2+1)^2} \, dx \). This has to be broken up into the sum of two terms the first of which is easy to handle, \( \frac{3}{2} \int \frac{2x}{(x^2+1)^2} \, dx \) and the second of which, \( -\int \frac{1}{(x^2+1)^2} \, dx \) needs...

So if the rational function is not in one of the above reduced forms, follow the following 4 steps:
1. If the polynomial in the numerator (numerator/denominator) has a degree greater than or equal to the one in the denominator then first divide the denominator into the numerator to obtain a polynomial plus a remainder where the remainder is a rational function which has the same denominator that you started with but the numerator now has a lower degree than the denominator. Examples:

\[
\frac{2x^4 + 6x^3 - 10x^2 - 3x + 2}{x^3 + 3x^2 - x - 3} = 2x + \frac{2x^2 + 13x + 2}{x^3 + 3x^2 - x - 3}.
\]

\[\int \frac{x^2 + 1}{x^2 - 3x + 2} \, dx\]

Remember - the degree of a polynomial is the highest power of \(x\) which has a non-zero coefficient. In the last example the polynomial in the numerator has degree 4 and the one in the denominator has degree 3.

2. Factor the denominator into irreducible polynomials. Each irreducible factor will be either linear, i.e., of the form \(ax + b\) or quadratic, i.e., of the form \(ax^2 + bx + c\). Remember in general factoring is not easy, but you should be familiar with the tricks of the trade. For instance:

\[x^2 - a^2 = (x - a)(x + a)\]

which gives \(x^3 - 4x = x(x - 2)(x + 2)\).

Use the quadratic formula if necessary to determine if a quadratic is irreducible or not:

\[x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}\].

If \(b^2 - 4ac\) is negative then the quadratic is irreducible. For example:

\[4x^2 + 1\] gives \(b^2 - 4ac = -16\) and so is irreducible.

3. Form the partial fraction decomposition of the integral which has the polynomial in the numerator of lower degree than the one in the denominator.

(a) Look at each factor of the denominator obtained in step 2. Make sure you have collected factors which are the same and express them as the factor to a power.

For example: \((x + 1)(x + 1)\) becomes \((x + 1)^2\). Now, there are 2 different types of factors, linear factors raised to some power, perhaps just the power of 1, and irreducible quadratic factors raised to some power. Each factor raised to the power \(n\) determines the denominators of \(n\) terms in the partial fraction decomposition. The numerators in these terms are unknowns to be determined in the next step. For instance, a factor of \((2x + 4)^3\) gives 3 terms:

\[\frac{A}{2x + 4} + \frac{B}{(2x + 4)^2} + \frac{C}{(2x + 4)^3}\]

and a factor of \((x^2 + 1)^2\) gives rise to 2 terms:

\[\frac{Ax + B}{x^2 + 1} + \frac{Cx + D}{(x^2 + 1)^2}\]

If the factor is linear, the numerators will be constants. If the factor is an irreducible quadratic then the numerator must be linear. Make sure that you give
different names to each unknown constant. For example if the polynomial is factored as \((2x + 4)^2(x^2 + 1)\) the the partial fraction decomposition is:

\[
\frac{A}{2x + 4} + \frac{B}{(2x + 4)^2} + \frac{Cx + D}{x^2 + 1}
\]

(b) Now suppose that we are integrating the function: \(\frac{2x^2 - 3x + 7}{(2x + 4)^2(x^2 + 1)}\). Then the next step is to solve for the unknown constants \(A, B, C\) and \(D\). Let

\[
\frac{2x^2 - 3x + 7}{(2x + 4)^2(x^2 + 1)} = \frac{A}{2x + 4} + \frac{B}{(2x + 4)^2} + \frac{Cx + D}{x^2 + 1}.
\]

Then,

\[
2x^2 - 3x + 7 = A(2x + 4)(x^2 + 1) + B(x^2 + 1) + (Cx + D)(2x + 4)^2
\]

\[
= A(2x^3 + 2x + 4x^2 + 4) + B(x^2 + 1) + Cx(4x^2 + 16x + 16) + D(4x^2 + 16x + 16).
\]

Collecting the coefficients of each power of \(x\), we get:

\[
2x^2 - 3x + 7 = (2A + 4C)x^3 + (4A + B + 16C + 4D)x^2 + (2A + 16C + 16D)x + (4A + B + 16D).
\]

This gives 4 equations in 4 unknowns. The above example may be one of the most difficult types you will encounter in your homework. The equations are obtained by equating the coefficients of the polynomials on each side of the last equation.

1.) \(2A + 4C = 0\)
2.) \(4A + B + 16C + 4D = 2\)
3.) \(2A + 16C + 16D = -3\)
4.) \(4A + B + 16D = 7\)

Solve for \(A, B, C\) and \(D\): From equation 1, \(A = -2C\). Then equation 3 gives,

\[
12C + 16D = -3 \Rightarrow D = (-3 - 12C)/16.
\]

Then from equation 2 we have,

\[-8C + B + 16C - 3/4 - 3C = 2 \Rightarrow B + 5C = 11/4\]

and from equation 4 we have,

\[-8C + B - 3 - 12C = 7 \Rightarrow B - 20C = 10.\]

Subtracting the last 2 equations we get \(25C = 129/4\), so

\[
C = -29/100, \quad A = 29/50, \quad D = 3/100, \quad B = 21/5.
\]

4. Finally, integrate the reduced forms of rational functions.
Summary

To integrate a quotient of polynomials, the following steps are suggested.

step 1 Perform polynomial long division, if necessary, to reduce the problem to integrating a rational function whose numerator has degree less than the degree of the denominator.

step 2 Factor the denominator into linear and irreducible quadratic factors.

step 3 Obtain the partial fraction decomposition of the rational function.

step 4 Integrate the summands in the resulting decomposition of the original rational function.

Exercises

1. \( \int \frac{x-1}{x^2} \, dx \)
2. \( \int \frac{x^5-4x^3+2x^2-7}{x^4} \, dx \)
3. \( \int \frac{x^3-2x}{x+1} \, dx \)
4. \( \int \frac{x^4+3x^2}{2x-3} \, dx \)
5. \( \int \frac{x^2-x}{x^2+1} \, dx \)
6. \( \int \frac{1}{x^2-x} \, dx \)
7. \( \int \frac{x^3+4}{x^2-4} \, dx \)
8. \( \int \frac{x^3+x^2-x+8}{x^2+2x} \, dx \)
9. \( \int \frac{x^4-2x^2+6}{x^2-3x+2} \, dx \)
10. \( \int \frac{x^4-x^2-4x-2}{x^3-x} \, dx \)
11. \( \int \frac{3x+4}{x^3-2x^2-3x} \, dx \)