16. The curves $y = x$ and $y = x^n$ cross at $x = 0$ and $x = 1$. For $0 < x < 1$, the curve $y = x$ is above $y = x^n$. Thus the area is given by

$$A_n = \int_0^1 (x - x^n) \, dx = \left[ \frac{x^2}{2} - \frac{x^{n+1}}{n+1} \right]_0^1 = \frac{1}{2} - \frac{1}{n+1} \to \frac{1}{2}.$$ 

Since $x^n \to 0$ for $0 \leq x < 1$, as $n \to \infty$, the area between the curves approaches the area under the line $y = x$ between $x = 0$ and $x = 1$.

17. Triangle of base and height 1 and 3. See Figure 8.7. (Either 1 or 3 can be the base. A non-right triangle is also possible.)

18. Semicircle of radius $r = 9$. See Figure 8.8.

19. Quarter circle of radius $r = \sqrt{15}$. See Figure 8.9.
20. Triangle of base and height 7 and 5. See Figure 8.10. (Either 7 or 5 can be the base. A non-right triangle is also possible.)

![Triangle Diagram](image)

21. Hemisphere with radius 12. See Figure 8.11.

![Hemisphere Diagram](image)

22. Cone with height 12 and radius 12/3 = 4. See Figure 8.12.

![Cone Diagram](image)

23. Cone with height 6 and radius 3. See Figure 8.13.

![Cone Diagram](image)
16. 

Radius = $\frac{b\sqrt{a^2 - z^2}}{a}$

$$y^2 = b^2 \left(1 - \frac{x^2}{a^2}\right).$$

$$V = \int_{-a}^{a} \pi y^2 \, dx = \pi \int_{-a}^{a} b^2 \left(1 - \frac{x^2}{a^2}\right) \, dx$$

$$= 2\pi b^2 \int_{0}^{a} \left(1 - \frac{x^2}{a^2}\right) \, dx = 2\pi b^2 \left[x - \frac{x^3}{3a^2}\right]_0^a$$

$$= 2\pi b^2 \left(a - \frac{a^3}{3a^2}\right) = 2\pi b^2 \left(a - \frac{1}{3}a\right)$$

$$= \frac{4}{3} \pi a b^2.$$

We slice the region perpendicular to the $x$-axis. The Riemann sum we get is $\sum \pi (x^3 + 1) \Delta x$. So the volume $V$ is the integral

$$V = \int_{-1}^{1} \pi (x^3 + 1)^2 \, dx$$

$$= \pi \int_{-1}^{1} (x^6 + 2x^3 + 1) \, dx$$

$$= \pi \left(\frac{x^7}{7} + \frac{x^4}{4} + x\right) \bigg|_{-1}^{1}$$

$$= (16/7) \pi \approx 7.18.$$

17.

Radius = $1 + x^3$

$$x$$

We slice the region perpendicular to the $x$-axis. The Riemann sum we get is $\sum \pi (x^3 + 1)^2 \Delta x$. So the volume $V$ is the integral

$$V = \int_{-1}^{1} \pi (x^3 + 1)^2 \, dx$$

$$= \pi \int_{-1}^{1} (x^6 + 2x^3 + 1) \, dx$$

$$= \pi \left(\frac{x^7}{7} + \frac{x^4}{4} + x\right) \bigg|_{-1}^{1}$$

$$= (16/7) \pi \approx 7.18.$$

18.

Radius = $1 - x$

$(x = 1)$

We slice the region perpendicular to the $y$-axis. The Riemann sum we get is $\sum \pi (1 - x)^2 \Delta y = \sum \pi (1 - y^2)^2 \Delta y$. So the volume $V$ is the integral

$$V = \int_{0}^{1} \pi (1 - y^2)^2 \, dy$$

$$= \pi \int_{0}^{1} (1 - 2y^2 + y^4) \, dy$$

$$= \pi \left(y - \frac{2y^3}{3} + \frac{y^5}{5}\right) \bigg|_{0}^{1}$$

$$= (8/15) \pi \approx 1.68.$$

19.

Radius = $\sin x$

We take slices perpendicular to the $x$-axis. The Riemann sum for approximating the volume is $\sum \pi \sin^2 x \Delta x$. The volume is the integral corresponding to that sum, namely

$$V = \int_{0}^{\pi} \pi \sin^2 x \, dx$$

$$= \pi \left[-\frac{1}{2} \sin x \cos x + \frac{1}{2} x\right]_0^\pi = \frac{\pi^2}{2} \approx 4.935.$$
16. (a) Divide the atmosphere into spherical shells of thickness $\Delta h$. See Figure 8.31. The density on a typical shell, $\rho(h)$, is approximately constant. The volume of the shell is approximately the surface area of a sphere of radius $r_e + h$ meters times $\Delta h$, where $r_e = 6.4 \cdot 10^6$ meters is the radius of the earth,

$$\text{Volume of Shell} \approx 4\pi (r_e + h)^2 \Delta h.$$

A Riemann sum for the total mass is

$$\text{Mass} \approx \sum \left(4\pi (r_e + h)^2 \times 1.28e^{-0.000124h} \Delta h \right) \text{ kg}.$$

(b) This Riemann sum becomes the integral

$$\text{Mass} = 4\pi \int_0^{100} (r_e + h)^2 \cdot 1.28e^{-0.000124h} \, dh$$

$$= 4\pi \int_0^{100} (6.4 \cdot 10^6 + h)^2 \cdot 1.28e^{-0.000124h} \, dh.$$

Evaluating the integral using numerical methods gives $M = 6.5 \cdot 10^{16}$ kg.

17. We need the numerator of $\bar{x}$, to be zero, i.e. $\sum x_i m_i = 0$. Since all of the masses are the same, we can factor them out and write $4 \sum x_i = 0$. Thus the fourth mass needs to be placed so that all of the positions sum to zero. The first three positions sum to $(-6 + 1 + 3) = -2$, so the fourth mass needs to be placed at $x = 2$.

18. We have

$$\text{Total mass of the rod} = \int_0^3 (1 + x^2) \, dx = \left[ x + \frac{x^3}{3} \right]_0^3 = 12 \text{ grams}.$$

In addition,

$$\text{Moment} = \int_0^3 x(1 + x^2) \, dx = \left[ \frac{x^2}{2} + \frac{x^4}{4} \right]_0^3 = \frac{99}{4} \text{ gram-meters}.$$

Thus, the center of mass is at the position $\bar{x} = \frac{99/4}{12} = 2.06$ meters.
10. Let $x$ be the distance measured from the bottom the tank. To pump a layer of water of thickness $\Delta x$ at $x$ feet from the bottom, the work needed is

$$(62.4)\pi 6^2(20 - x)\Delta x.$$ 

Therefore, the total work is

$$W = \int_{0}^{10} 36 \cdot (62.4)\pi (20 - x)dx$$

$$= 36 \cdot (62.4)\pi (20x - \frac{1}{2}x^2)\bigg|_{0}^{10}$$

$$= 36 \cdot (62.4)\pi (200 - 50)$$

$$\approx 1,058,591.1 \text{ ft-lb}.$$ 

11. Let $x$ be the distance from the bottom of the tank. To pump a layer of water of thickness $\Delta x$ at $x$ feet from the bottom to 10 feet above the tank, the work done is $(62.4)\pi 6^2(30 - x)\Delta x$. Thus the total work is

$$\int_{0}^{20} 36 \cdot (62.4)\pi (30 - x)dx$$

$$= 36 \cdot (62.4)\pi \left(30x - \frac{1}{2}x^2\right)\bigg|_{0}^{20}$$

$$= 36 \cdot (62.4)\pi (30(20) - \frac{1}{2}20^2)$$

$$\approx 2,822,909.50 \text{ ft-lb}.$$ 

Volume of Slice = $\pi 6^2 \Delta x$
6. Since the function levels off at the value of $c$, the area under the graph is not finite, so it is not 1. Thus, this function cannot be a pdf.

It is a cdf and $c = 1$. The cdf is given by

$$P(x) = \begin{cases} 
0 & \text{for } x < 0 \\
\frac{x}{5} & \text{for } 0 \leq x \leq 5 \\
1 & \text{for } x > 5.
\end{cases}$$

The pdf in Figure 8.59 is given by

$$p(x) = \begin{cases} 
0 & \text{for } x < 0 \\
\frac{1}{5} & \text{for } 0 \leq x \leq 5 \\
0 & \text{for } x > 5.
\end{cases}$$

7. This function decreases, so it cannot be a cdf. Since the graph must represent a pdf, the area under it is 1. The region consists of two rectangles, each of base 0.5, and one of height $2c$ and one of height $c$, so

$$\text{Area} = 2c(0.5) + c(0.5) = 1$$

$$c = \frac{1}{1.5} = \frac{2}{3}$$

The pdf is therefore

$$p(x) = \begin{cases} 
0 & \text{for } x < 0 \\
\frac{4}{3} & \text{for } 0 \leq x \leq 0.5 \\
\frac{2}{3} & \text{for } 0.5 < x \leq 1 \\
0 & \text{for } x > 1.
\end{cases}$$

The cdf $P(x)$ is the antiderivative of this function with $P(0) = 0$. See Figure 8.60. The formula for $P(x)$ is

$$P(x) = \begin{cases} 
0 & \text{for } x < 0 \\
\frac{4x}{3} & \text{for } 0 \leq x \leq 0.5 \\
\frac{2}{3} + \left(\frac{2}{3}\right)(x - 0.5) & \text{for } 0.5 < x \leq 1 \\
1 & \text{for } x > 1.
\end{cases}$$
16. (a) The area under the graph of the height density function $p(x)$ is concentrated in two humps centered at 0.5 m and 1.1 m. The plants can therefore be separated into two groups, those with heights in the range 0.3 m to 0.7 m, corresponding to the first hump, and those with heights in the range 0.9 m to 1.3 m, corresponding to the second hump. This grouping of the grasses according to height is probably close to the species grouping. Since the second hump contains more area than the first, there are more plants of the tall grass species in the meadow.

(b) As do all cumulative distribution functions, the cumulative distribution function $P(x)$ of grass heights rises from 0 to 1 as $x$ increases. Most of this rise is achieved in two spurts, the first as $x$ goes from 0.3 m to 0.7 m, and the second as $x$ goes from 0.9 m to 1.3 m. The plants can therefore be separated into two groups, those with heights in the range 0.3 m to 0.7 m, corresponding to the first spurt, and those with heights in the range 0.9 m to 1.3 m, corresponding to the second spurt. This grouping of the grasses according to height is the same as the grouping we made in part (a), and is probably close to the species grouping.

(c) The fraction of grasses with height less than 0.7 m equals $P(0.7) = 0.25 = 25\%$. The remaining 75\% are the tall grasses.

17. (a) The percentage of calls lasting from 1 to 2 minutes is given by the integral
\[ \int_{1}^{2} p(x) \, dx \int_{1}^{2} 0.4e^{-0.4x} \, dx = e^{-0.4} - e^{-0.8} \approx 22.1\%. \]

(b) A similar calculation (changing the limits of integration) gives the percentage of calls lasting 1 minute or less as
\[ \int_{0}^{1} p(x) \, dx = \int_{0}^{1} 0.4e^{-0.4x} \, dx = 1 - e^{-0.4} \approx 33.0\%. \]

(c) The percentage of calls lasting 3 minutes or more is given by the improper integral
\[ \int_{3}^{\infty} p(x) \, dx = \lim_{b \to \infty} \int_{3}^{b} 0.4e^{-0.4x} \, dx = \lim_{b \to \infty} (e^{-1.2} - e^{-0.4b}) = e^{-1.2} \approx 30.1\%. \]

(d) The cumulative distribution function is the integral of the probability density; thus,
\[ C(h) = \int_{0}^{h} p(x) \, dx = \int_{0}^{h} 0.4e^{-0.4x} \, dx = 1 - e^{-0.4h}. \]

18. (a) The fraction of students passing is given by the area under the curve from 2 to 4 divided by the total area under the curve. This appears to be about $\frac{2}{3}$.

(b) The fraction with honor grades corresponds to the area under the curve from 3 to 4 divided by the total area. This is about $\frac{1}{3}$.

(c) The peak around 2 probably exists because many students work to get just a passing grade.

(d) Most of the earth's surface is below sea level. Much of the earth's surface is either around 3 miles below sea level or exactly at sea level. It appears that essentially all of the surface is between 4 miles below sea level and 2 miles above sea level. Very little of the surface is around 1 mile below sea level.

(b) The fraction below sea level corresponds to the area under the curve from $-4$ to 0 divided by the total area under the curve. This appears to be about $\frac{3}{4}$.
2. Recall that the mean is \( \int_{-\infty}^{\infty} xp(x) \, dx \). In the fishing example, \( p(x) = 0 \) except when \( 2 \leq x \leq 8 \), so the mean is

\[
\int_{2}^{8} xp(x) \, dx.
\]

Using the equation for \( p(x) \) from the graph,

\[
\int_{2}^{8} xp(x) \, dx = \int_{2}^{6} xp(x) \, dx + \int_{6}^{8} xp(x) \, dx
\]

\[
= \int_{2}^{6} x(0.04x) \, dx + \int_{6}^{8} x(-0.06x + 0.6) \, dx
\]

\[
= \frac{0.04x^3}{3} \bigg|_{2}^{6} + \left( -0.02x^3 + 0.3x^2 \right) \bigg|_{6}^{8}
\]

\[
\approx 5.253 \text{ tons}.
\]

3. (a) (i) \( p(x) \)

(b) Recall that the mean is the "balancing point." In other words, if the area under the curve was made of cardboard, we'd expect it to balance at the mean. All of the graphs are symmetric across the line \( x = \mu \), so \( \mu \) is the "balancing point" and hence the mean.

As the graphs also show, increasing \( \sigma \) flattens out the graph, in effect lessening the concentration of the data near the mean. Thus, the smaller the \( \sigma \) value, the more data is clustered around the mean.

Problems

4. (a) Since \( d(e^{-ct})/dt = ce^{-ct} \), we have

\[
c \int_{0}^{6} e^{-ct} \, dt = -e^{-ct} \bigg|_{0}^{6} = 1 - e^{-6c} = 0.1,
\]

so

\[
c = \frac{-1}{6} \ln 0.9 \approx 0.0176.
\]

(b) Similarly, with \( c = 0.0176 \), we have

\[
c \int_{6}^{12} e^{-ct} \, dt = -e^{-ct} \bigg|_{6}^{12} = e^{-6c} - e^{-12c} = 0.9 - 0.81 = 0.09,
\]

so the probability is 9%.