MTH 513
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Definition. Let $V$ be a vector space over $F$. An inner product on $V$ is a function that assigns to every ordered pair of vectors $x$ and $y$ in $V$, a scalar in $F$, denoted $\langle x, y \rangle$, such that for all $x, y,$ and $z$ in $V$ and all $c$ in $F,$ the following hold:

(a) $\langle x + z, y \rangle = \langle x, y \rangle + \langle z, y \rangle$.
(b) $\langle cx, y \rangle = c \langle x, y \rangle$.
(c) $\langle x, y \rangle = \overline{\langle y, x \rangle}$, where the bar denotes complex conjugation.
(d) $\langle x, x \rangle > 0$ if $x \neq 0$.

A vector space $V$ over $F$ endowed with a specific inner product is called an inner product space.

Theorem 6.1. Let $V$ be an inner product space. Then for $x, y, z \in V$ and $c \in F$, the following statements are true.

(a) $\langle x, y + z \rangle = \langle x, y \rangle + \langle x, z \rangle$.
(b) $\langle x, cy \rangle = c \langle x, y \rangle$.
(c) $\langle x, 0 \rangle = \langle 0, x \rangle = 0$.
(d) $\langle x, x \rangle = 0$ if and only if $x = 0$.
(e) If $\langle x, y \rangle = \langle x, z \rangle$ for all $x \in V$, then $y = z$.

Proof.

(a) Using part (c) of definition, take conjugate; then, by part (a),

$\langle x, y + z \rangle = \overline{\langle y + z, x \rangle} = \overline{\langle y, x \rangle + \langle z, x \rangle} = \overline{\langle y, x \rangle} + \overline{\langle z, x \rangle}$

Taking conjugate again,

$\overline{\overline{\langle x, y \rangle + \langle x, z \rangle}} = \langle x, y \rangle + \langle x, z \rangle$.

(b) Take conjugate

$\langle x, cy \rangle = \overline{\langle cy, x \rangle}$

Using part (b) of definition

$\overline{c \langle y, x \rangle} = c \overline{\langle y, x \rangle} = c \langle y, x \rangle$. 
(c) \( <x, 0> = <x, 0x> \)

By part (b) of theorem,

\[
0 = 0 <x, x> \quad \Rightarrow \quad 0 = 0
\]

\[
= 0.
\]

Also,

\[
<x, x> = <0x, x> = 0 <x, x> = 0.
\]

(d) Assume \( x = 0 \)

By (c) of theorem,

\[
<x, 0> = 0.
\]

Assume \( <x, x> = 0 \)

By part (d) of definition, if \( x \neq 0 \), then \( <x, x> > 0 \).

By way of contradiction, assume \( <x, x> = 0 \) and \( x \neq 0 \).

This is impossible, according to the definition, so

\( x = 0 \).

(e) \( <x, y> - <x, z> = 0 \)

\[
\text{THEOREM}
\]

By part (a) of definition,

\[
<x, y - z> = 0 \quad \text{Must hold for all } x, \text{ even for } x = y - z.
\]

By part (d) of theorem,

\[
<x, y - z> = 0 \iff y - z = 0.
\]

Hence, \( y = z \).