Chapter 1

Defn 1. Field Axioms. A set $S$ with operations $+$ and $\cdot$ and distinguished elements $0$ and $1$ with $0 \neq 1$ is a field if the following properties hold for all $x, y, z \in S$.

A0: $x + y \in S$. Closure of addition.
M0: $x \cdot y \in S$. Closure of multiplication.
A1: $(x + y) + z = x + (y + z)$. Associativity of addition.
M1: $(x \cdot y) \cdot z = x \cdot (y \cdot z)$. Associativity of multiplication.
A2: $x + y = y + x$. Commutativity of addition.
M2: $x \cdot y = y \cdot x$. Commutativity of multiplication.
A3: $x + 0 = x$. Additive identity.
M3: $x \cdot 1 = x$. Multiplicative identity.
A4: Given $x$, there is an $w \in S$ such that $x + w = 0$. Additive inverse.
M4: Given $x \neq 0$, there is an $w \in S$ such that $x \cdot w = 1$. Multiplicative inverse.
DL: $x \cdot (y + z) = x \cdot y + x \cdot z$. Distributive Law.

The operations $+$ and $\cdot$ are called addition and multiplication. The elements 0 and 1 are the additive identity element and the multiplicative identity element. We assume additionally that “=” is an equivalence relation, that is, $a = a$, if $a = b$ then $b = a$, and if $a = b$ and $b = c$, then $a = c$.

Defn 2. A vector space $V$ over a field $F$ consists of a set on which two operations, which are called addition and scalar multiplication are defined so that for each pair of elements $x, y \in V$, there is a unique element $x + y \in V$, and for each element $a$ in $F$ and each element $x$ in $V$, there is a unique element $a \cdot x$ in $V$, such that the following conditions hold.

(VS 1) For all $x, y \in V$, $x + y = y + x$ commutativity of addition.
(VS 2) For all $x, y, z \in V$, $(x + y) + z = x + (y + z)$ associativity of addition.
(VS 3) There exists an element in $V$ denoted by 0 such that $x + 0 = x$ for each $x \in V$.
(VS 4) For each element $x \in V$, there exists an element $y \in V$ such that $x + y = 0$.
(VS 5) For each $x \in V$, $1x = x$.
(VS 6) For each pair of elements $a, b \in F$ and each element $x \in V$, $(ab)x = a(bx)$.
(VS 7) For each element $a$ in $F$ and each pair of elements $x, y \in V$, $a(x + y) = ax + ay$.
(VS 8) For each pair of elements $a, b \in F$, and each element $x \in V$, $(a + b)x = ax + bx$.

Theorem 1.3. Let $V$ be a vector space and $W$ a subset of $V$. Then $W$ is a subspace of $V$ if and only if the following three conditions hold for the operations defined in $V$.

1. $0 \in W$.
2. $x + y \in W$ whenever $x \in W$ and $y \in W$.
3. $c \cdot x \in W$ whenever $c \in F$ and $x \in W$.

Defn 3. Let $V$ be a vector space of $F$. A linear combination is of the form $a_1v_1 + a_2v_2 + \cdots + a_nv_n$. We call $a_i$ for all $i \in \{1, \ldots, n\}$ the coefficients.

Defn 4. Let $S \subseteq V$. The span of $S$, span($S$) = \{ $a_1v_1 + a_2v_2 + \cdots + a_nv_n \in S$ \}. By convention, span($\emptyset$) = \{ $0$ \}. Also note that $S \subseteq \text{span}(S)$.

Theorem 1.5. The span of any subset $S$ of a vector space $V$ is a subspace of $V$. Moreover, any subspace of $V$ that contains $S$ must also contain the span of $S$.

Cor 1. If $S$ is a subspace of $V$, then span($S$) = $S$. 

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Defn 5. A set \( S \) generates a vector space \( V \) if \( \text{span}(S) = V \).

Example 1. \( \mathbb{R}^2 = V \) and \( \{(1,0),(0,1)\} = S \).

Defn 6. A subset \( S \) of a vector space \( V \) is called linearly dependent if there exists a finite number of distinct scalars and vectors, not all zero, such that

\[ a_1v_1 + \cdots + a_nv_n = 0. \]

Also, we say that \( \{v_1,\ldots,v_n\} \) is linearly dependent. If no such set exists, then \( S \) is called linearly independent.

Theorem 1.6. Let \( V \) be a vector space, and let \( S_1 \subseteq S_2 \subseteq V \). If \( S_1 \) is linearly dependent, then \( S_2 \) is linearly dependent. Also, if \( S_2 \) is linearly independent, then \( S_1 \) is linearly independent.

Cor 2. Let \( V \) be a vector space, and let \( S_1 \subseteq S_2 \subseteq V \). If \( S_2 \) is linearly independent, then \( S_1 \) is linearly independent.

Theorem 1.7. Let \( S \) be a linearly independent subset of a vector space \( V \), and let \( v \) be a vector in \( V \) that is not in \( S \). Then \( S \cup \{v\} \) is linearly dependent if and only if \( v \in \text{span}(S) \).

Defn 7. A basis \( \beta \) for a vector space \( V \) is a linearly independent subset of \( V \) that generates \( V \). If \( \beta \) is a basis for \( V \), we also say that the vectors of \( \beta \) form a basis for \( V \).

Theorem 1.8. Let \( V \) be a vector space and \( \beta = \{u_1,u_2,\ldots,u_n\} \) be a subset of \( V \). Then \( \beta \) is a basis for \( V \) if and only if each \( v \in V \) can be uniquely expressed as a linear combination of vectors of \( \beta \), that is, can be expressed in the form

\[ v = a_1u_1 + a_2u_2 + \cdots + a_nu_n \]

for unique scalars \( a_1,a_2,\ldots,a_n \).

Theorem 1.9. If a vector space \( V \) is generated by a finite set \( S \), then some subset of \( S \) is a basis for \( V \). Hence \( V \) has a finite basis.

Theorem 1.10. (Replacement Theorem). Let \( V \) be a vector space that is generated by a set \( G \) containing exactly \( n \) vectors, and let \( \mathcal{L} \) be a linearly independent subset of \( V \) containing exactly \( m \) vectors. Then \( m \leq n \) and there exists a subset \( H \) of \( G \) containing exactly \( n - m \) vectors such that \( \mathcal{L} \cup H \) generates \( V \).

Cor 3. (1, To Theorem 1.10) Let \( V \) be a vector space having a finite basis. Then every basis for \( V \) contains the same number of vectors.

Defn 8. A vector space \( V \) is called \textit{finite-dimensional} if it has finite basis. The size of which is called the dimension of \( V \), denoted \( \dim(V) \). A vector space that is not finite-dimensional is called \textit{infinite-dimensional}.

Cor 4. (2, To Theorem 1.10) Let \( V \) be a vector space with dimension \( n \).

(1) Any finite generating set of \( V \) contains at least \( n \) vectors, and a generating set for \( V \) that contains exactly \( n \) vectors is a basis for \( V \).

(2) Any linearly independent subset of \( V \) that contains exactly \( n \) vectors is a basis for \( V \).

(3) Every linearly independent subset of \( V \) can be extended to a basis for \( V \).

Theorem 1.11. Let \( W \) be a subspace of a finite-dimensional vector space \( V \). Then \( W \) is finite-dimensional and \( \dim(W) \leq \dim(V) \). Moreover, if \( \dim(W) = \dim(V) \), then \( V = W \).

Cor 1. (to Theorem 1.11) If \( W \) is a subspace of a finite-dimensional vector space \( V \), then any basis for \( W \) can be extended to a basis for \( V \).
Chapter 2

Defn 9. (p. 65) Let $V$ and $W$ be vector spaces (over $F$). We call a function $T : V \to W$ a linear transformation form $V$ to $W$ if, for all $x, y \in V$ and $c \in F$, we have

(a) $T(x + y) = T(x) + T(y)$ and
(b) $T(cx) = cT(x)$.

Fact 1. (p. 65)

1. If $T$ is linear, then $T(0) = 0$.
2. $T$ is linear if and only if $T(cx + y) = cT(x) + T(y)$ for all $x, y \in V$ and $c \in F$.
3. If $T$ is linear, then $T(x - y) = T(x) - T(y)$ for all $x, y \in V$.
4. $T$ is linear if and only if,
   for $x_1, x_2, \ldots, x_n \in V$ and $a_1, a_2, \ldots, a_n \in F$, we have
   
   \[ T \left( \sum_{i=1}^{n} a_i x_i \right) = \sum_{i=1}^{n} a_i T(x_i). \]

Defn 10. (p. 67) Let $V$ and $W$ be vector spaces, and let $T : V \to W$ be linear. We define the null space (or kernel) $N(T)$ of $T$ to be the set of all vectors $x$ in $V$ such that $T(x) = 0$; $N(T) = \{ x \in V : T(x) = 0 \}$.

We define the range (or image) $R(T)$ of $T$ to be the subset $W$ consisting of all images (under $T$) of vectors in $V$; that is, $R(T) = \{ T(x) : x \in V \}$.

Theorem 2.1. Let $V$ and $W$ be vector spaces and $T : V \to W$ be linear. Then $N(T)$ and $R(T)$ are subspaces of $V$ and $W$, respectively.

Theorem 2.2. Let $V$ and $W$ be vector spaces, and let $T : V \to W$ be linear. If $\beta = \{ v_1, v_2, \ldots, v_n \}$ is a basis for $V$, then

\[ R(T) = \text{span}(T(\beta)) = \text{span}(\{ T(v_1), T(v_2), \ldots, T(v_n) \}). \]

Defn 11. (p. 69) Let $V$ and $W$ be vector spaces, and let $T : V \to W$ be linear. If $N(T)$ and $R(T)$ are finite-dimensional, then we define the nullity of $T$, denoted $\text{nullity}(T)$, and the rank of $T$, denoted $\text{rank}(T)$, to be the dimensions of $N(T)$ and $R(T)$, respectively.

Theorem 2.3. (Dimension Theorem). Let $V$ and $W$ be vector spaces, and let $T : V \to W$ be linear. If $V$ is finite-dimensional, then $\text{nullity}(T) + \text{rank}(T) = \text{dim}(V)$.

Theorem 2.4. Let $V$ and $W$ be vector spaces, and let $T : V \to W$ be linear. Then $T$ is one-to-one if and only if $N(T) = \{ 0 \}$.

Theorem 2.5. Let $V$ and $W$ be vector spaces of equal (finite) dimension, and let $T : V \to W$ be linear. Then the following are equivalent.

(a) $T$ is one-to-one.
(b) $T$ is onto.
(c) $\text{rank}(T) = \text{dim}(V)$.

Theorem 2.6. Let $V$ and $W$ be vector spaces over $F$, and suppose that $\{ v_1, v_2, \ldots, v_n \}$ is a basis for $V$. For $w_1, w_2, \ldots, w_n$ in $W$, there exists exactly one linear transformation $T : V \to W$ such that $T(v_i) = w_i$ for $i = 1, 2, \ldots, n$.

Cor 1. Let $V$ and $W$ be vector spaces, and suppose that $V$ has a finite basis $\{ v_1, v_2, \ldots, v_n \}$. If $U, T : V \to W$ are linear and $U(v_i) = T(v_i)$ for $i = 1, 2, \ldots, n$, then $U = T$. 

Defn 12. (p. 80) Let $\beta = \{u_1, u_2, \ldots, u_n\}$ be an ordered basis for a finite-dimensional vector space $V$. For $x \in V$, let $a_1, a_2, \ldots, a_n$ be the unique scalars such that

$$x = \sum_{i=1}^{n} a_i u_i$$

We define the coordinate vector of $x$ relative to $\beta$, denoted $[x]_\beta$, by

$$[x]_\beta = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{pmatrix}.$$ 

Defn 13. (p. 80) Let $V$ and $W$ be finite-dimensional vector spaces with ordered bases $\beta = \{v_1, v_2, \ldots, v_n\}$ and $\gamma = \{w_1, w_2, \ldots, w_m\}$, respectively. Let $T : V \rightarrow W$ be linear. Then for each $j$, $1 \leq j \leq n$, there exists unique scalars $a_{i,j} \in F$, $a \leq i \leq m$, such that

$$T(v_j) = \sum_{i=1}^{m} a_{i,j} w_i \quad 1 \leq j \leq n.$$ 

We call the $m \times n$ matrix $A$ defined by $A_{i,j} = a_{i,j}$ the matrix representation of $T$ in the ordered bases $\beta$ and $\gamma$ and write $A = [T]_\beta^\gamma$. If $V = W$ and $\beta = \gamma$, then we write $A = [T]_\beta$.

Fact 2. $[a_1 x_1 + a_2 x_2 + \cdots + a_n x_n]_B = a_1 [x_1]_B + a_2 [x_2]_B + \cdots + a_n [x_n]_B$.

Fact 2. (a) Let $V$ and $W$ be finite dimensional vector spaces with bases, $\beta$ and $\gamma$, respectively. Then

$$[T]_\beta^\gamma [x]_\beta = [T(x)]_\gamma.$$ 

Fact 3. Let $V$ be a vector space with basis $B$. $\{x_1, x_2, \ldots, x_k\}$ is linearly independent if and only if $\{[x_1]_B, [x_2]_B, \ldots, [x_k]_B\}$ is linearly independent.

Defn 14. (p. 99) Let $V$ and $W$ be vector spaces, and let $T : V \rightarrow W$ be linear. A function $U : W \rightarrow V$ is said to be an inverse of $T$ if $TU = I_W$ and $UT = I_V$. If $T$ has an inverse, then $T$ is said to be invertible.

Fact 4. If $T$ is invertible, then the inverse of $T$ is unique.

Defn 15. We denote the inverse of $T$ by $T^{-1}$.

Theorem 2.17. Let $V$ and $W$ be vector spaces, and let $T : V \rightarrow W$ be linear. Then $T^{-1} : W \rightarrow V$ is linear.

Fact 5. If $T$ is a linear transformation between vector spaces of equal (finite) dimension, then the conditions of being invertible, one-to-one, and onto are all equivalent.

Defn 16. (p. 100) Let $A$ be an $n \times n$ matrix. Then $A$ is invertible if there exits an $n \times n$ matrix $B$ such that $AB = BA = I$.

Fact 6. If $A$ is invertible, then the inverse of $A$ is unique.

Defn 17. We denote the inverse of $A$ by $A^{-1}$.

Lemma 1. (p. 101) Let $T$ be an invertible linear transformation from $V$ to $W$. Then $V$ is finite-dimensional if and only if $W$ is finite-dimensional. In this case, dim$(V) = $ dim$(W)$.
Theorem 2.18. Let $V$ and $W$ be finite-dimensional vector spaces with ordered bases $\beta$ and $\gamma$, respectively. Let $T : V \to W$ be linear. Then $T$ is invertible if and only if $[T]_\beta$ is invertible. Furthermore, $[T^{-1}]_\gamma = ([T]_\beta)^{-1}$.

Cor 1. Let $V$ be a finite-dimensional vector space with an ordered basis $\beta$, and let $T : V \to V$ be linear. Then $T$ is invertible if and only if $[T]_\beta$ is invertible. Furthermore, $[T^{-1}]_\beta = ([T]_\beta)^{-1}$.

Fact 6. (a) $L_A$ is a linear transformation and for $\beta = \{e_1, e_2, \ldots, e_n\}$, $[L_A]_\beta = A$.

Cor 2. Let $A$ be an $n \times n$ matrix. Then $A$ is invertible if and only if $L_A$ is invertible. Furthermore, $(L_A)^{-1} = L_A^{-1}$.

Defn 18. Let $V$ and $W$ be vector spaces. We say that $V$ is isomorphic to $W$ if there exists a linear transformation $T : V \to W$ that is invertible. Such a linear transformation is called and isomorphism from $V$ onto $W$.

Theorem 2.19. Let $V$ and $W$ be finite-dimensional vector spaces (over the same field). Then $V$ is isomorphic to $W$ if and only if $\dim(V) = \dim(W)$.

Fact 7. Let $P \in M_n(F)$ be invertible. $W$ is a subspace of $F^n$ implies $L_P(W)$ is a subspace of $F^n$ and $\dim(L_P(W)) = \dim(W)$.

Fact 8. Let $S \in M_n(F)$. If $S$ is invertible, then $R(S) = F^n$.

Fact 9. $S, T \in M_n(F)$, $S$ invertible and $T$ invertible imply $ST$ is invertible and its inverse is $T^{-1}S^{-1}$.

Theorem 2.20. Let $V$ and $W$ be finite-dimensional vector spaces over $F$ of dimensions $n$ and $m$, respectively, and let $\beta$ and $\gamma$ be ordered bases for $V$ and $W$, respectively. Then the function

$$\Phi : \mathcal{L}(V, W) \to M_{m \times n}(F)$$

defined by $\Phi(T) = [T]_\beta^\gamma$ for $T \in \mathcal{L}(V, W)$, is an isomorphism.

Cor 1. Let $V$ and $W$ be finite-dimensional vector spaces over $F$ of dimensions $n$ and $m$, respectively. Then $\mathcal{L}(V, W)$ is finite-dimensional of dimension $mn$.

Theorem 2.22. Let $\beta$ and $\beta'$ be two ordered bases for a finite-dimensional vector space $V$, and let $Q = [I_V]^{\beta'}_\beta$. Then

(a) $Q$ is invertible.

(b) For any $v \in V$, $[v]_\beta = Q[v]_{\beta'}$.

Defn 19. (p. 112) The matrix $Q = [I_V]^{\beta'}_\beta$ defined in Theorem 2.22 is called a change of coordinate matrix. Because of part (b) of the theorem, we say that $Q$ changes $\beta'$-coordinates into $\beta$-coordinates. Notice that $Q^{-1} = [I_V]^{\beta}_{\beta'}$.

Defn 20. (p. 112) A linear transformation $T : V \to V$ is called a linear operator.

Theorem 2.23. Let $T$ be a linear operator on a finite-dimensional vector space $V$, and let $\beta$ and $\beta'$ be ordered bases for $V$. Suppose that $Q$ is the change of coordinate matrix that changes $\beta'$-coordinates into $\beta$-coordinates. Then

$$[T]_{\beta'} = Q^{-1}[T]_{\beta}Q.$$
Cor 1. Let $A \in M_{n \times n}(F)$, and let $\gamma$ be an ordered basis for $F^n$. Then $[L_A]_{\gamma} = Q^{-1}AQ$, where $Q$ is the $n \times n$ matrix whose $j^{th}$ column is the $j^{th}$ vector of $\gamma$.

Defn 21. (p. 115) Let $A$ and $B$ be in $M_n(F)$. We say that $B$ is similar to $A$ if there exists an invertible matrix $Q$ such that $B = Q^{-1}AQ$.

Theorem 2.24. Suppose that $V$ is a finite-dimensional vector space with the ordered basis $\beta = \{x_1, x_2, \ldots, x_n\}$. Let $f_i (1 \leq i \leq n)$ be the $i^{th}$ coordinate function with respect to $\beta$ and let $\beta^* = \{f_1, f_2, \ldots, f_n\}$. Then $\beta^*$ is an ordered basis for $V^*$, and, for any $f \in V^*$, we have

$$f = \sum_{i=1}^{n} f(x_i)f_i.$$