Theorem 1. (1.3) Let \( V \) be a vector space and \( W \) a subset of \( V \). Then \( W \) is a subspace of \( V \) if and only if the following three conditions hold for the operations defined in \( V \).

1. \( 0 \in W \).
2. \( x + y \in W \) whenever \( x \in W \) and \( y \in W \).
3. \( cs \in W \) whenever \( c \in F \) and \( x \in W \).

Theorem from first set of notes.

1. \( \det A \) can be evaluated by expanding along any row or column, that is, \( \det A \) equals the sum of the products of the entries along any row or column with their cofactors.
   - This part was left as an exercise: For all \( n \in \mathbb{N} \), if \( A \in M_n(F) \) then “expansion along row 1" is equivalent to “expansion along column 1".
2. \( \det A = \det A^t \).
3. If any row or column = 0 then \( \det A = 0 \).
4. For any \( c \) and \( n \times n \) \( A \), \( \det(cA) = c^n \det A \).
5. If \( A \) has two rows that are equal (or columns) then \( \det A = 0 \).
6. If \( A' \) is produced from \( A \) by interchanging two rows (or two columns) then \( \det A' = -\det A \).
7. If \( A' \) is produced from \( A \) by replacing a row (column) or \( A \) with a constant times that row (column) then \( \det A' = c \det A \).
8. If \( A' \) is produced from \( A \) by replacing one row (column) with that row (column) plus some multiple of a different row (column) then \( \det A' = \det A \).

From second set of notes.

Theorem 2. Let \( E_{ij}, E_i(c), \) and \( E_{ij}(c) \) denote the elementary matrices. Then

1. \( \det E_{ij} = -1 \)
2. \( \det E_i(c) = c \)
3. \( \det E_{ij}(c) = 1 \)
4. If \( E \) is any \( n \times n \) elementary matrix and \( A \) is any \( n \times n \) matrix, then

\[
\det(EA) = \det E \det A
\]

and by induction, we obtain for elementary matrices \( E_1, E_2, \ldots, E_r \),

\[
\det(E_1E_2\cdots E_rA) = \det E_1 \det E_2 \cdots \det E_r \det A
\]

Theorem 3. \( A \in M_n(F) \)

1. \( A \) is invertible \( \iff \) \( \det A \neq 0 \)
2. If \( A \) is invertible, then \( \det(A^{-1}) = 1/\det A \)

Theorem 4. \( \det(AB) = \det A \det B \)
From Sections 1.4, 1.5, 1.6.

**Theorem 5.** (1.5) The span of any subset $S$ of a vector space $V$ is a subspace of $V$. Moreover, any subspace of $V$ that contains $S$ must also contain the span of $S$.

**Theorem 6.** (1.6) Let $V$ be a vector space, and let $S_1 \subseteq S_2 \subseteq V$. If $S_1$ is linearly dependent, then $S_2$ is linearly dependent. Also, if $S_2$ is linearly independent, then $S_1$ is linearly independent.

**Theorem 7.** (1.7) Let $S$ be a linearly independent subset of a vector space $V$, and let $v$ be a vector in $V$ that is not in $S$. Then $S \cup \{v\}$ is linearly dependent if and only if $v \in \text{span}(S)$.

**Theorem 8.** (1.8) Let $V$ be a vector space and $\beta = \{u_1, u_2, \ldots, u_n\}$ be a subset of $V$. Then $\beta$ is a basis for $V$ if and only if each $v \in V$ can be uniquely expressed as a linear combination of vectors of $\beta$, that is, can be expressed in the form

$$v = a_1u_1 + a_2u_2 + \cdots + a_nu_n$$

for unique scalars $a_1, a_2, \ldots, a_n$.

**Theorem 9.** (1.9) If a vector space $V$ is generated by a finite set $S$, then some subset of $S$ is a basis for $V$. Hence $V$ has a finite basis.

**Theorem 10.** (1.10) (Replacement Theorem). Let $V$ be a vector space that is generated by a set $G$ containing exactly $n$ vectors, and let $L$ be a linearly independent subset of $V$ containing exactly $m$ vectors. Then $m \leq n$ and there exists a subset $H$ of $G$ containing exactly $n - m$ vectors such that $L \cup H$ generates $V$.

**Cor 1.** (1, To Theorem 1.10) Let $V$ be a vector space having a finite basis. Then every basis for $V$ contains the same number of vectors.

**Cor 2.** (2, To Theorem 1.10) Let $V$ be a vector space with dimension $n$.

1. Any finite generating set of $V$ contains at least $n$ vectors, and a generating set for $V$ that contains exactly $n$ vectors is a basis for $V$.
2. Any linearly independent subset of $V$ that contains exactly $n$ vectors is a basis for $V$.
3. Every linearly independent subset of $V$ can be extended to a basis for $V$.

**Theorem 11.** (1.11) Let $W$ be a subspace of a finite-dimensional vector space $V$. Then $W$ is finite-dimensional and $\dim(W) \leq \dim(V)$. Moreover, if $\dim(W) = \dim(V)$, then $V = W$.

**Cor 3.** (to Theorem 1.11) If $W$ is a subspace of a finite-dimensional vector space $V$, then any basis for $W$ can be extended to a basis for $V$.

From 3rd set of notes:

**Theorem 12.** Given $V$ and $W$, vector spaces,

$$T : V \to W$$

is a linear transformation if and only if for all $x, y \in V$, $a, b \in F$,

$$T(ax + by) = aT(x) + bT(y).$$

**Theorem 13.** Given $A \in M_{m,n}(F)$, $V$ an $n$ dimensional vector space with basis $v_1, v_2, \ldots, v_n$. $W$ an $m$ dimensional vector space with basis $w_1, w_2, \ldots, w_m$. There exists a linear transformation $T$ such that $A$ represents $T$. Conversely, given $T$, there is an $A$. 
Fact 1. If $A$ is invertible, then the only solution to $Ax = 0$ is $x = 0$.

Claim 1. The set of invertible matrices in $M_n$ is the set
$$\{B_2[I]_{B_1} : B_1, B_2 \text{ are bases of } F^n\}$$

Claim 2. The set of all basis representations of $T$ is
$$\{B[I]_{B[T]_{B_1}}B[I]_{B_1} : B_1 \text{ is a basis of } V\}$$
$$= \{S[T]_{B}S^{-1} : S \text{ is invertible}\}$$

Fact 2. If $A$ is similar to $B$ then $\det(A) = \det(B)$.

Back to the Book:

Theorem 14. (2.3) (Dimension Theorem). Let $V$ and $W$ be vector spaces, and let $T : V \to W$ be linear. If $V$ is finite-dimensional, then $\text{nullity}(T) + \text{rank}(T) = \dim(V)$.

From Class notes.

Fact 3. (A) If $S$ is an invertible $n \times n$ matrix then $R(S) = F^n$.

Fact 4. (B) $S, T$ invertible $\Rightarrow ST$ invertible.

Fact 5. (C) $[a_1x_1 + a_2x_2 + \cdots + a_nx_n]_B = a_1[x_1]_B + a_2[x_2]_B + \cdots + a_n[x_n]_B$

Fact 6. (D) Let $V$ be a vector space with basis $B$. $\{x_1, x_2, \ldots, x_k\}$ is linearly independent if and only if $\{[x_1]_B, [x_2]_B, \ldots, [x_k]_B\}$ is linearly independent.

Fact 7. (E) Let $P \in M_n(F)$, invertible. $W$ is a subspace of $F^n$ implies $L_P(W)$ is a subspace of $F^n$ and $\dim(L_P(W)) = \dim(W)$.

Back to the Book.

Theorem 15. (2.19) Let $V$ and $W$ be finite-dimensional vector spaces (over the same field). Then $V$ is isomorphic to $W$ if and only if $\dim(V) = \dim(W)$.

Theorem 16. (3.3) Let $T : V \to W$ be a linear transformation between finite-dimensional vector spaces, and let $\beta$ and $\gamma$ be ordered bases for $V$ and $W$, respectively. Then $\text{rank}(T) = \text{rank}(\gamma[T]_{\beta})$.

Theorem 17. (3.4) Let $A$ be an $m \times n$ matrix. If $P$ and $Q$ are invertible $m \times m$ and $n \times n$ matrices, respectively, then

1. $\text{rank}(AQ) = \text{rank}(A)$,
2. $\text{rank}(PA) = \text{rank}(A)$,
3. and $\text{rank}(PAQ) = \text{rank}(A)$.

Cor 4. (To Theorem 3.4) Elementary row and column operations on a matrix are rank-preserving.

Theorem 18. (3.5) The rank of any matrix equals the maximum number of its linearly independent columns; that is, the rank of a matrix is the dimension of the subspace generated by its columns.