Mth 142    Quiz 10 (on 7.3, 7.4, 7.5)    Summer 2005

Name:

1. Solve the following differential equation using the separation of variables technique.

\[ \int y \, dy = \int x \, dx \]
\[ \frac{dy}{dx} = \frac{x}{y} \]

\[ \frac{1}{2} y^2 = \frac{1}{2} x^2 + C_1 \]
\[ y^2 = x^2 + C_2 \]
\[ y = \pm \sqrt{x^2 + C_2} \]

2. The temperature, \( H \), of a hot steel ingot placed in a room held at 70 degrees changes at each moment at a rate proportional to the difference between the ingot temperature and the room temperature. When the ingot is first placed in the room, its temperature is 800 degrees. One half hour later its temperature is 300 degrees.

\[
\begin{array}{c|c}
0 & 800 \\
\frac{1}{2} & 300 \\
\end{array}
\]

(a) Write a differential equation satisfied by \( H \). Use the letter \( k \) for the constant of proportionality in the equation.

\[ \frac{\Delta H}{\Delta t} = k \left( H - 70 \right) \]

(b) Give an explicit formula for \( H \) as a function of time, resolving the unknown constant and \( k \).

Using \((0, 800)\) \[ H = 70 + B e^{kt} \]

\[ 800 = 70 + B \]
\[ 730 = B \]

Using \((\frac{1}{2}, 300)\) \[ y = 300 = 70 + 730e^{-\frac{1}{2}k} \]
\[ 230 = 730e^{-\frac{1}{2}k} \]
\[ k = \ln \left( \frac{230}{730} \right) \]

\[ k = 0.5 \]
(c) Rewrite the differential equation, giving values for any constants.

\[ \frac{dT}{dt} = -2.3099 (T - 70) \]

(d) Sketch a slope field for your equation showing any equilibrium solutions.

(e) Use (b) to determine the time at which the ingot will reach 200 degrees.

\[ 200 = 70 + 730 e^{-2.3099 t} \]

\[ \frac{130}{730} = e^{-2.3099 t} \]

\[ \frac{\ln(\frac{130}{730})}{-2.3099} = t \]

\[ t = 0.747 \text{ hrs.} \]

\[ \approx \frac{3}{4} \text{ hr} \]
3. In the 1930s, the Soviet ecologist G.F. Gause studied the population growth of yeast. Fit a logistic curve,

$$\frac{dP}{dt} = kP(1 - P/L), \quad (1)$$

to his data below using the method outlined below.

<table>
<thead>
<tr>
<th>Time (hours)</th>
<th>0</th>
<th>10</th>
<th>18</th>
<th>23</th>
<th>34</th>
<th>42</th>
<th>47</th>
</tr>
</thead>
<tbody>
<tr>
<td>Yeast population</td>
<td>.37</td>
<td>8.87</td>
<td>10.66</td>
<td>12.50</td>
<td>13.27</td>
<td>12.87</td>
<td>12.70</td>
</tr>
</tbody>
</table>

(a) Plot the data and use it to estimate (by eye) the carrying capacity, $L$.

![Plot of yeast population growth]

$L = 13$

(b) Use the first two pieces of data in the table, and your value of $L$, to estimate $k$.

The slope $\frac{dP}{dt} \approx \frac{8.87 - .37}{10 - 0}$ for $P$ somewhere between $P = .37$ and $P = 8.87$. Let's say $P = 2.5$. Then $\frac{.85}{2.5} = k \approx (1 - \frac{2.5}{13})$

$k \approx 42$

(c) Using your values for $k$ and $L$, find the general solution for the logistic equation (1) with initial population $P_0 = .37$.

$$P = \frac{L}{1 + Ae^{-kt}} = \frac{13}{1 + A e^{-42t}}$$

where $A = \frac{L - P_0}{P_0} = \frac{13 - .37}{.37} = 34.1351$

(d) Sketch the solution curve on the same axes as the data points.

See above.