**Theorem 5.23 (Cayley-Hamilton)**
Let $T$ be a linear operator on a finite-dimensional vector space $V$, and let $f(t)$ be the characteristic polynomial of $T$. Then $f(T)=T_0$, the zero transformation.

**Corollary to Theorem 5.23 (Cayley-Hamilton Theorem for Matrices)**
Let $A$ be an $n \times n$ matrix, and let $f(x)$ be the characteristic polynomial of $A$. Then $f(A)=O$, the $n \times n$ zero matrix.

**Definition.** Let $f(x)=a_0+a_1x+\ldots+a_nx^n$ be a polynomial with coefficients from a field $F$. If $T$ is a linear operator on a vector space $V$ over $F$, we define
\[
 f(T) = a_0I_V + a_1T + \ldots + a_nT^n
\]
Similarly, if $A$ is a $n \times n$ matrix with entries from $F$, we define
\[
 f(A) = a_0I + a_1A + \ldots + a_nA^n
\]

**Proof.** Since $f(x)$ is the characteristic polynomial of $A$, by definition, $f(x)$ is also the characteristic polynomial of the linear operator $L_A$.
Let $f(x)=a_0+a_1x+\ldots+a_nx^n$, then $f(L_A)=a_0I + a_1L_A + \ldots + a_nL_A^n$.

For any $\nu \in F^n$, $f(L_A)(\nu) = (a_0I + a_1L_A + \ldots + a_nL_A^n)(\nu)$
\[
 = a_0\nu + a_1(A+ \ldots + a_nA^n)\nu
 = (a_0I + a_1A + \ldots + a_nA^n)\nu
\]
Notice that $f(L_A)=T_0$ by Theorem 5.23. So
\[
 (a_0I + a_1A + \ldots + a_nA^n)\nu = 0
\]
Since $\nu$ is arbitrary, we know $a_0I + a_1A + \ldots + a_nA^n=O$, the $n \times n$ zero matrix.
Exercise 5 in section 5.4
Let $T$ be a linear operator on a vector space $V$. Prove that the intersection of any collection of $T$-invariant subspaces of $V$ is a $T$-invariant subspace of $V$.

Proof. We only need to show that the intersection of two $T$-invariant subspaces of $V$ is a $T$-invariant subspace of $V$.

Let $W_1$ and $W_2$ be two $T$-invariant subspaces of $V$, and let $W$ be the intersection of $W_1$ and $W_2$.

First we will prove that $W$ is $T$-invariant.

For any $v \in W$, we also know $v \in W_1$ and $v \in W_2$. Since $W_1$ and $W_2$ are $T$-invariant, $T(v) \in W_1$ and $T(v) \in W_2$. So $T(v) \in W_1 \cap W_2 = W$. This proves that $W$ is $T$-invariant.

Then we will show that $W$ is a subspace of $V$.

Since $0 \in W_1$ and $0 \in W_2$, $0 \in W$.

For any $x, y \in W$, since $x, y \in W_1$, $x, y \in W_2$ and $W_1$ and $W_2$ are subspaces of $V$, we know $x + y \in W_1$ and $x + y \in W_2$. So $x + y \in W_1 \cap W_2 = W$.

For any $x \in W$ and scalar $c$, since $x \in W_1$, $x \in W_2$ and $W_1$ and $W_2$ are subspaces of $V$, we know $cx \in W_1$ and $cx \in W_2$. So $cx \in W_1 \cap W_2 = W$.

By Theorem 1.3, $W$ is a subspace of $V$. This completes the proof.