Covers: 2.4, Focus on Theory, 3.1, 3.2, 3.3, 3.4, 3.5, Focus on Practice, 4.1

(1) Let \( f(z) = 3z^2 + 4z + 1 \).
(a) Express \( f'(z) \) as a limit.
\[
\begin{align*}
\text{Sol:} & \quad f'(z) = \lim_{h \to 0} \frac{3(z + h)^2 + 4(z + h) + 1 - (3z^2 + 4z + 1)}{h} \\
& \quad = \lim_{h \to 0} \frac{3z^2 + 6zh + 3h^2 + 4z + 4h + 1 - 3z^2 - 4z - 1}{h} \\
& \quad = \lim_{h \to 0} \frac{6zh + 3h^2 + 4h}{h} \\
& \quad = \lim_{h \to 0} (6z + 3h + 4) \\
& \quad = 6z + 4 
\end{align*}
\]
(b) Using algebra and the algebraic definition of the derivative, evaluate the limit in part (a) to show that \( f'(z) = 6z + 4 \).
\[
\begin{align*}
\text{Sol:} & \quad \text{The Numerator:} \\
& \quad = 3(z + h)^2 + 4(z + h) + 1 - (3z^2 + 4z + 1) \\
& \quad = 3z^2 + 6zh + 3h^2 + 4z + 4h + 1 - 3z^2 - 4z - 1 \\
& \quad = 6zh + 3h^2 + 4h \\
& \quad = h(6z + 3h + 4) \\
& \quad \text{So,} \\
& \quad f'(z) = \lim_{h \to 0} \frac{h(6z + 3h + 4)}{h} \\
& \quad = \lim_{h \to 0} 6z + 3h + 4 \\
& \quad = 6z + 4 
\end{align*}
\]

(2) Let \( g(x) = \ln x \).
(a) Express \( g'(20) \) as a limit.
\[
\begin{align*}
\text{Sol:} & \quad g'(20) = \lim_{h \to 0} \frac{\ln(20 + h) - \ln(20)}{h} \\
& \quad \text{Estimate the limit in part (a) to estimate the value of } g'(20). \text{ Be sure to show your work.} \\
& \quad \begin{array}{c|c}
  h & \frac{\ln(20+h)-\ln(20)}{h} \\
  \hline
  -1 & .0501254 \\
  -.01 & .0500125 \\
  -.001 & .0500013 \\
  .001 & .0499988 \\
  .01 & .0499875 \\
  .1 & .0498756 \\
\end{array} \\
& \quad \text{So,} \\
& \quad g'(20) = \lim_{h \to 0} \frac{\ln(20 + h) - \ln(20)}{h} \approx .05 
\end{align*}
\]
Figure 1. Figure for problem number 3

(c) Check your answer using the "nDer" function, or any other derivative function, on your graphing calculator.

Sol: On the TI-86, the syntax is: “nDer(ln x,x,20)”

(3) Sketch a graph of
\[ f(x) = \frac{e^x - 1}{x} \]

(4) Fill in the following table with values of \( f(x) = \frac{e^x - 1}{x} \).

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>-0.1</td>
<td>0.9516258</td>
</tr>
<tr>
<td>-0.01</td>
<td>0.9950166</td>
</tr>
<tr>
<td>-0.001</td>
<td>0.9995002</td>
</tr>
<tr>
<td>0.001</td>
<td>1.0005</td>
</tr>
<tr>
<td>0.01</td>
<td>1.005017</td>
</tr>
<tr>
<td>0.1</td>
<td>1.051709</td>
</tr>
</tbody>
</table>

(5) Use the values you found in the previous problem, estimate

\[ \lim_{x \to 0} \frac{e^x - 1}{x} \]

Sol: \( \approx 1 \).
(6) For each point, fill in the table with +, −, or 0 according to whether $f(x)$, $f'(x)$, and $f''(x)$ is positive, negative, or zero.

<table>
<thead>
<tr>
<th>Point</th>
<th>P1</th>
<th>P2</th>
<th>P3</th>
<th>P4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(x)$</td>
<td>−</td>
<td>+</td>
<td>+</td>
<td>+</td>
</tr>
<tr>
<td>$f'(x)$</td>
<td>+</td>
<td>0</td>
<td>−</td>
<td>+</td>
</tr>
<tr>
<td>$f''(x)$</td>
<td>−</td>
<td>−</td>
<td>0</td>
<td>+</td>
</tr>
</tbody>
</table>

Figure 2. Figure for problem number 6

(7) For the graph below, name the intervals of $x$ for which,
(a) The first derivative is positive.
(b) The second derivative is negative.

Figure 3. Figure for problem number 7

Sol:
(a) $-1 \leq x \leq 2$
(b) $.5 \leq x \leq 3$
Some of the values of \( g(x) \) are given in the following table.

<table>
<thead>
<tr>
<th>( x )</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>( g(x) )</td>
<td>2.24</td>
<td>2.45</td>
<td>2.65</td>
<td>2.83</td>
<td>3.00</td>
<td>3.16</td>
</tr>
</tbody>
</table>

(a) Estimate the values of \( g'(3) \) and \( g'(4) \).

(b) Estimate the value of \( g''(3.5) \).

\[
\begin{align*}
\text{Sol:} & \\
(a) & \quad g'(3) \approx \frac{g(4) - g(2)}{4 - 2} = \frac{3 - 2.65}{2} = .175 \\
& \quad g'(4) \approx \frac{g(5) - g(3)}{5 - 3} = \frac{3.16 - 2.83}{2} = .165 \\
(b) & \quad g''(3.5) \approx \frac{g''(4) - g''(3)}{4 - 3} = \frac{.165 - .175}{1} = -.01
\end{align*}
\]

9) The function \( f(x) \) has the form \( f(x) = ax^2 + bx + c \). We know that \( f(-2) = 0 \), \( f'(-2) = 6 \), and \( f''(-2) = 4 \). Find the values of \( a \), \( b \), and \( c \).

\[
\begin{align*}
\text{Sol:} & \\
& \quad f(x) = ax^2 + bx + c \Rightarrow f(-2) = a(-2)^2 + b(-2) + c = 0 \Rightarrow 4a - 2b + c = 0 \\
& \quad f'(x) = 2ax + b \Rightarrow f'(-2) = 2a(-2) + b = 6 \Rightarrow -4a + b = 6 \\
& \quad f''(x) = 2a \Rightarrow f''(-2) = 2a = 4 \Rightarrow a = 2 \\
& \quad -4a + b = 6 \Rightarrow -8 + b = 6 \Rightarrow b = 14 \\
& \quad 4a - 2b + c = 0 \Rightarrow 8 - 28 + c = 0 \Rightarrow c = 20
\end{align*}
\]

10) Find \( \frac{dy}{dx} \) when \( y = x + 3 - \sqrt{x} + 4x^3 \).

\[
\begin{align*}
\text{Sol:} & \\
& \quad y = x + 3 - x^{1/2} + 4x^3 \\
& \quad y' = 1 - \frac{1}{2}x^{-1/2} + 12x^2
\end{align*}
\]

11) (a) Let \( f(x) = \sin x - \cos x \). Find \( f'(x) \).

\[
\text{Sol:} f'(x) = \cos x + \sin x
\]

(b) Let \( g(t) = \sin(4t) \). Find \( g'(t) \).

\[
\text{Sol:} g'(t) = 4 \cos(4t)
\]

12) (a) Let \( f(x) = \ln x - 3^x \). Find \( f'(x) \).

\[
\text{Sol:} f'(x) = \frac{1}{x} - \ln 3 \cdot 3^t
\]

(b) Let \( g(t) = \ln(t^2) \). Find \( g'(t) \).

\[
\text{Sol:} g'(t) = \frac{1}{t^2} \cdot 2t = \frac{2}{t}
\]
Suppose that \( t \) weeks after the end of an advertising campaign the weekly sales of a certain item are \( 100(8t - t^2) \). At what instantaneous rate are the sales changing when \( t \) equals 6? Name the label on this rate.

\[
S(t) = 100(8t - t^2) \\
S'(t) = 100(8 - 2t) = 800 - 200t \\
S'(6) = 800 - 1200 = -400
\]

items per week. This means that in the 6th week the sales are dropping at the rate of 400 items per week. So the 6th week sales are 400 less than the 5th week sales (approximately).

(14) Evaluate the following derivatives.

(a) Find \( f'(t) \) when \( f(t) = e^{-t^2+1} \).

\[
Sol: f'(t) = (-2t)e^{-t^2+1}
\]

(b) Find \( g'(s) \) when \( g(s) = \frac{s}{\ln s} \).

\[
Sol: g'(s) = \frac{\ln s - 1}{(\ln s)^2}
\]

(15) Let \( f(x) = 6x^2 - 5x + 2 \). Find \( f'(0) \) and the equation of the tangent line at \((0, f(0))\).

\[
Sol: f'(x) = 12x - 5, \quad f'(0) = -5, \quad \text{and} \quad f(0) = 2. \quad \text{The line with slope -5 and through point (0, 2) is:} \quad y = -5x + 2.
\]

(16) Let \( f(x) = -8x^3 + 6x - 1 \).

(a) Find all critical points of \( f(x) \).

\[
Sol: f'(x) = -24x^2 + 6, \quad f'(x) = 0, \quad \text{for} \quad x = -0.5, 0.5.
\]

(b) Use the 2\textsuperscript{nd} derivative test to find all local extreme values of \( f(x) \).

\[
Sol: f''(x) = -48x, \quad f''(-0.5) \text{ is positive, so } f \text{ is concave up at } x = -0.5, \quad \text{so there is a local minimum at } x = -0.5. \quad f''(0.5) \text{ is negative, so } f \text{ is concave down at } x = 0.5, \quad \text{so there is a local maximum at } x = 0.5.
\]
(17) Let $f'(x) = (x - 1)(x - 3)(x - 5)$. Note this is the derivative of $f(x)$.

(a) Find all critical points of $f(x)$.

**Sol:** $x = 1, 3, 5$

(b) Use the 1st derivative test to determine the $x$ coordinates of all local minima and local maxima of $f(x)$.

**Sol:** Test $x = 0$ which is less than 1: $f'(0)$ negative,
Test $x = 2$ which is between 1 and 3: $f'(2)$ is positive,
Test $x = 4$ which is between 3 and 5: $f'(4)$ is negative,
Test $x = 6$ which is bigger than 5: $f'(6)$ is positive.
The result is local min at $x = 1$, local max at $x = 3$, local min at $x = 5$.

(18) Find derivatives for the following functions

(a) $f(x) = 5^x - 5x^4$

**Sol:** $f'(x) = \ln 5 \cdot 5^x - 20x^3$.

(b) $g(x) = x^3(e^{-4x} - 5)$

**Sol:** $g'(x) = (3x^2)(e^{-4x} - 5) + x^3(-4)e^{-4x}$

(c) $h(x) = 5\sin(x + 3^x)$

**Sol:** $h'(x) = 5(1 + \ln 3 \cdot 3^x)\cos(x + 3^x)$

(d)

$$p(x) = \ln\frac{x}{x^2 + 1}$$

**Sol:**

$$p'(x) = \frac{\frac{1}{x}(x^2 + 1) - \ln x(2x)}{(x^2 + 1)^2}$$

(e) $q(x) = \ln(\cos(x))$

**Sol:**

$$q'(x) = \frac{1}{\cos x}(-\sin x)$$

(f)

$$r(x) = \sqrt{2x + 1} - \frac{1}{\sqrt{2x + 1}}$$

**Sol:** $r(x) = (2x + 1)^{1/2} - (2x + 1)^{-1/2}$,

$$r'(x) = \frac{1}{2}(2x + 1)^{-1/2}(2) + \frac{1}{2}(2x + 1)^{-3/2}(2),$$

so $r'(x) = (2x + 1)^{-1/2} + (2x + 1)^{-3/2}$.
(19) Draw a possible graph of $y = f(x)$ given the following information about its derivative.

- $f'(x) > 0$ for $x < 2$
- $f'(x) < 0$ for $x > 2$
- $f''(x) > 0$ for $x < 0$.
- $f''(x) < 0$ for $x > 0$. 

**Figure 4.** Figure for problem number 19