Defn 1. (p. 152) If $A \in M_{m \times n}(F)$, we define the **rank** of $A$, denoted $\text{rank}(A)$, to be the rank of the linear transformation $L_A : F^n \to F^m$.

**Fact 10.** Let $V$ be a vector space of dimension $n$ over $F$ with basis $\beta$. The transformation $f : V \to F^n$ defined by $f(x) = [x]_\beta$ is an isomorphism.

Proof. Suppose $\beta = \{v_1, v_2, \ldots, v_n\}$ and recall the definition of $[x]_\beta$ given on page 80.

$f$ is one-to-one: Let $[x]_\beta = [y]_\beta$. Then there is a unique representation of $x$ as a linear combination of basis vectors: $x = a_1v_1 + a_2v_2 + \cdots + a_nv_n$, see Theorem 1.8. Likewise, $y$ has a unique representation $y = b_1v_1 + b_2v_2 + \cdots + b_nv_n$. But $[x]_\beta = [y]_\beta$ implies $(a_1, a_2, \ldots, a_n) = (b_1, b_2, \ldots, b_n)$. Thus, $x = y$.

$f$ is onto: Let $y \in F^n$, then, $y = (y_1, y_2, \ldots, y_n)$. For $x = y_1v_1 + y_2v_2 + \cdots + y_nv_n$, we have that $y = [x]_\beta$ and so $f(x) = y$.

This establishes a one-to-one correspondence between $V$ and $F^n$, which have already been established to be isomorphic by Theorem 2.19.

**Theorem 3.3.** Let $T : V \to W$ be a linear transformation between finite-dimensional vector spaces, and let $\beta$ and $\gamma$ be ordered bases for $V$ and $W$, respectively. Then $\text{rank}(T) = \text{rank}([T]_{\beta\gamma})$.

Proof. Let $\dim V = n$ and $\dim W = m$ and remember that the underlying field $F$ is the same for both $V$ and $W$. We have that $R(T) \subseteq W$ and

$$L_{[T]_{\beta\gamma}} : F^n \to F^m.$$ 

Then $R(L_{[T]_{\beta\gamma}}) \subseteq F^n$. We will show $R(T)$ is isomorphic to $R(L_{[T]_{\beta\gamma}})$, then by Theorem 2.19,

$$\dim(R(L_{[T]_{\beta\gamma}})) = \dim(R(T))$$

which implies that $\text{rank}(T) = \text{rank}(L_{[T]_{\beta\gamma}})$, which is equal to $\text{rank}([T]_{\beta\gamma})$ by definition.

Let $T(x) \in R(T)$, for some $x \in V$. Let $f : R(T) \to R(L_{[T]_{\beta\gamma}})$ be defined by

$$f(T(x)) = [T]_{\beta\gamma}[x]_\beta.$$ 

$f$ is one-to-one: Suppose $[T]_{\beta\gamma}[x]_\beta = [T]_{\beta\gamma}[y]_\beta$ then $T(x)_\gamma = T(y)_\gamma$ and by Fact 10, we have that $T(x) = T(y)$.

$f$ is onto: Let $[T]_{\beta\gamma}x \in R(L_{[T]_{\beta\gamma}})$ for some $x \in F^n$. Then, $x = [y]_\beta$ for some $y \in V$ and $[T]_{\beta\gamma}x = [T]_{\beta\gamma}[y]_\beta$ So $f(T(y)) = [T]_{\beta\gamma}[y]_\beta = [T]_{\beta\gamma}x$.

**Theorem 3.4.** Let $A$ be an $m \times n$ matrix. If $P$ and $Q$ are invertible $m \times m$ and $n \times n$ matrices, respectively, then

1. $\text{rank}(AQ) = \text{rank}(A)$,
2. $\text{rank}(PA) = \text{rank}(A)$,
3. and $\text{rank}(PAQ) = \text{rank}(A)$. 

1
Proof. Corollary 2 to Theorem 2.18 establishes that $L_Q$ is invertible. So in particular, $L_Q$ is onto and $L_Q(F^n) = F^n$. (a) Remember
\[
\text{rank}(AQ) = \dim(R(L_{AQ})).
\]
But $L_{AQ} = L_A \circ L_Q$. So $R(L_{AQ}) = L_A(L_Q(F^n))$. Then
\[
L_A(L_Q(F^n)) = L_A(F^n) = R(L_A).
\]
Hence, $\text{rank}(L_{AQ}) = \text{rank}(L_A)$ as needed.

(b) First, note that $\text{rank}(PA) = \text{rank}(L_{PA}).$

Again,
\[
L_{PA} = L_P \circ L_A
\]
and
\[
R(L_{PA}) = (L_P \circ L_A)(F^n) = L_P(L_A(F^n)).
\]
But, by Fact 7,
\[
\dim(L_P(L_A(F^n))) = \dim(L_A(F^n)).
\]
Then
\[
\text{rank}(PA) = \text{rank}(L_{PA}) = \dim(R(L_{PA})) = \dim(L_A(F^n)) = \dim(R(L_A)) = \text{rank}(L_A) = \text{rank}(A).
\]
(c) $\text{rank}(PAQ) = \text{rank}(AQ) = \text{rank}(A)$. \[\blacksquare\]

Cor 1. *(To Theorem 3.4)* Elementary row and column operations on a matrix are rank-preserving.

Proof. An elementary row operation on a matrix $A$ is equivalent to left multiplication by an elementary matrix, namely the one obtained by performing the same row operation on the identity.

An elementary column operation on a matrix $A$ is equivalent to right multiplication by an elementary matrix, namely the one obtained by performing the same column operation on the identity. \[\blacksquare\]

Theorem 3.5. *The rank of any matrix equals the maximum number of its linearly independent columns; that is, the rank of a matrix is the dimension of the subspace generated by its columns.*

Proof. Let $A \in M_{m \times n}(F)$. Suppose $S = \{A_1, A_2, \ldots, A_n\}$ is the set of columns of $A$. Then $S \subseteq F^m$.

Then
\[
\text{rank } A = \text{rank}(L_A) = \dim(R(L_A)).
\]
We want to show that $\text{span}(S) = R(L_A)$. Let $y \in R(L_A)$. Then, there exists $x \in F^n$ such that $Ax = y$; that is, for
\[
x = (x_1, \ldots, x_n)
\]
we have
\[
x_1A_1 + x_2A_2 + \cdots + x_nA_n = Ax = y.
\]
We have established that $\text{span}(S) \supseteq R(L_A)$.\]
On the other hand, a linear combination of the columns of $A$ $c_1A_1 + c_2A_2 + \cdots + c_nA_n$ can be interpreted as multiplication of $A$ times the coefficient vector $c = (c_1, c_2, \ldots, c_n)$, $c_1A_1 + c_2A_2 + \cdots + c_nA_n = Ac$. So, we have span($S$) $\subseteq$ R($A$).

\textbf{Defn 2.} The types of row operations are 1) interchanging rows, 2) multiplying a row by a constant, and 3) adding a scalar multiple of one row to another.

\textbf{Theorem 3.6.} Let $A$ be an $m \times n$ matrix of rank $r$. Then $r \leq m$, $r \leq n$, and, by means of a finite number of elementary row and column operations, $A$ can be transformed into the matrix

$$D = \begin{pmatrix} I_r & O_1 \\ O_2 & O_3 \end{pmatrix}$$

where $O_1$, $O_2$, and $O_3$ are zero matrices. Thus $D_{i,i} = 1$ for $i \leq r$ and $D_{i,j} = 0$ otherwise.

\textit{Proof.} If $A = 0$, then the result holds for $r = 0$.

Suppose $A \neq 0$ and $r = \text{rank } A > 0$. We will prove this case by induction on $m$.

Suppose $m = 1$, and $A = [a_1, a_2, \ldots, a_n]$ We produce a sequence of matrices $A, A_0, A_1, A_2, A_3, \ldots, A_n$ such that one is obtained from the last by an elementary row or column operation and $A_n = [1, 0, 0, \ldots, 0]$.

For each $\ell \in \{0, 1, \ldots, k\}$ and $j \in \{1, 2, \ldots, n\}$, we denote the $j^{th}$ column of $A_\ell$ by $a_{\ell,j}$. If $a_1 \neq 0$, we set $A_0 = A$. Otherwise, since $A \neq 0$, we know that for some $j \in [n]$, $a_j \neq 0$ and we interchange columns 1 and $j$ to form $A_0$.

To form $A_1$, we multiply row 1 of $A_0$ by $1/a_{0,1}$, so that column 1 of $A_1$ is equal to 1.

For $j \in \{2, 3, 4, \ldots, n\}$, we multiply column 1 of $A_{j-1}$ by $-a_{j-1,j}$ and add that to column $j$ of $A_{j-1}$ to obtain $A_j$, so that column $j$ of $A_j$ becomes $a_{j,j} = a_{j-1,j} - a_{j-1,j} = 0$.

Let $m > 1$ and assume that the conclusion holds for matrices with $m - 1$ rows. Let $A \in M_{m\times n}(F)$. We produce a sequence of matrices $A, A_0, A_1, A_2, A_3, \ldots, A_k$ such that one is obtained from the last by an elementary row or column operation and $A_n = D$.

We denote the entry in row $i$, column $j$ of $A$ by $a_{i,j}$ and for each $\ell \in \{0, 1, \ldots, k\}$, $j \in [n]$, and $i \in [m]$, we denote the $i^{th}$ row, $j^{th}$ column of $A_\ell$ by $a_{\ell,i,j}$. If $a_{1,1} \neq 0$, we set $A_0 = A$.

If not then for some $i \in [m]$, we know that there is a some $j \in [n]$ such that $a_{i,j} \neq 0$ we first switch row $i$ with row 1 of $A$ and then switch column $j$ with column 1 of the resulting matrix to form $A_0$. Thus $a_{0,1,1} \neq 0$.

To form $A_1$, we multiply row 1 of $A_0$ by $1/a_{0,1,1}$, so that entry $a_{1,1,1}$ of $A_1$ equals 1.

For $j \in \{2, 3, 4, \ldots, n\}$, we multiply column 1 of $A_{j-1}$ by $-a_{j-1,1,j}$ and add that to column $j$ of $A_{j-1}$ to obtain $A_j$, so that the first entry in column $j$ of $A_j$ becomes $a_{j,1,j} = a_{j-1,1,j} - a_{j-1,1,j} = 0$. In this way, we obtain that the first row of $A_n$ is 1, 0, 0, \ldots, 0.

Set $A_{n+1} = A_n$. Similarly, for $\ell \in \{n + 2, n + 3, \ldots, n + m\}$, and $i = \ell - n$, we multiply row 1 of $A_{\ell-1}$ by $-a_{\ell-1,1,i}$ and add that to row $i$ of $A_{\ell-1}$ to obtain $A_{\ell}$, so that the first entry in row $i$ of $A_{\ell}$ becomes $a_{\ell,i,1} = a_{\ell-1,1,i} - a_{\ell-1,1,i} = 0$ and for $j \in \{2, 3, \ldots, n\}$, the $j^{th}$ column of $A_{\ell}$ becomes $a_{\ell,j,i} = a_{\ell-1,j,i} - a_{\ell-1,j,i} \cdot 0 = a_{\ell-1,j,i}$. In this way, we obtain that the first column of $A_{n+m}$ is 1, 0, 0, \ldots, 0.

Using the notation that for a given $m \times n$ matrix $B$, $B(i|j)$ is the $m - 1 \times n - 1$ matrix obtained from $B$ by removing row $i$ and column $j$, we have that $B = A_{n+m}(1|1)$ is a
$m - 1 \times n - 1$ matrix and so by induction can be reduced by $t$ elementary row and column operations to the matrix

$$D(B) = \begin{pmatrix} I_k & O_1 \\ O_2 & O_3 \end{pmatrix}$$

where $k$ is the rank of $B$. Adjusting by $1$, we perform this same sequence of operations, starting with $A_{n+m}$, creating a sequence $A_{n+m}, A_{n+m+1}, \ldots, A_{n+m+t}$. That is to say row $i$ of $B$ is translated to row $i+1$ of $A_\ell$ and similarly column $j$ of $B$ is translated to column $j+1$ of $A_\ell$. As row 1 and column 1 of $A_\ell$ is never touched, we have that

$$A_{n+m+t} = \begin{pmatrix} 1 & 0 & 4 & 0 & 5 \\ 0 & 1 & 7 & 0 & 6 \\ 0 & 7 & O_2 & O_3 \end{pmatrix} = D$$

where $0_4$ is a row of $k$ zeros, $0_5$ is a row of $n - k + 1$ zeros, $0_6$ is a column of $k$ zeros, and $0_7$ is a column of $m - k + 1$ zeros.

The rank of $D$ is $k + 1$, as we know by examining the $k + 1$ linear independent columns and using Theorem 3.5. By the corollary to Theorem 3.4, the rank of $A$ is preserved by elementary row and column operations in producing $D$. Thus the rank of $A$ must also be $k + 1$. Thus $r = k + 1$.

**Cor 1.** Let $A$ be an $m \times n$ matrix of rank $r$. Then there exists invertible matrices $B \in M_m(F)$ and $C \in M_n(F)$ such that $D = BAC$, where

$$D = \begin{pmatrix} I_r & O_1 \\ O_2 & O_3 \end{pmatrix}$$

is the $m \times n$ matrix in which $O_1, O_2, \text{ and } O_3 \text{ are zero matrices.}$

**Proof.** Let $A \in M_{m,n}(F)$ be of rank $r$. Since the result $S$ of a row operation is the same as multiplying $A$ on the left by an elementary matrix to obtain $S$ and a column operation on $S$ is the same as multiplying $S$ on the right by an elementary matrix, we have two matrices, $B \in M_m(F)$ and $C \in M_n(F)$ such that

$$BAC = D = \begin{pmatrix} I_r & O_1 \\ O_2 & O_3 \end{pmatrix}$$

where $B$ is the product of the elementary matrices and $C$ is a product of elementary matrices, and thus both are invertible.

**Defn 3.** $A \in M_{m,n}(F), A^t \in M_{n,m}(F)$, such that entry $i, j$ of $A$ is entry $j, i$ of $A^t$.

**Fact 11.** $A \in M_{m,n}(F), B \in M_{n,m}(F)$, $(AB)^t = B^tA^t$. If $A \in M_n(F)$ and $A$ is invertible, then $A^t$ is invertible.

**Cor 2.** Let $A \in M_{m,n}(F)$. Then

(a) $\text{rank}(A^t) = \text{rank}(A)$.

(b) The rank of any matrix equals the maximum number of its linearly independent rows; that is, the rank of a matrix is the dimension of the subspace generated by its rows.

(c) The rows and columns of any matrix generate subspaces of the same dimension, numerically equal to the rank of the matrix.
Proof. (a) By Theorem 3.6, there exists invertible matrices $B$ and $C$ such that $A = BDC$ and

$$D = \begin{pmatrix} I_r & O_1 \\ O_2 & O_3 \end{pmatrix}.$$ 

By Theorem 3.4, rank $D = \text{rank}(BDC) = \text{rank} A = r$. Also, $A^t = C^tD^tB^t$. By Theorem 3.4, rank $D^t = \text{rank}(C^tD^tB^t) = \text{rank} A^t$. Clearly rank $D^t = r$, so we have, rank $A = \text{rank} A^t$.

(b) By Theorem 3.5, the rank of $A^t$ is the dimension of its column space. But the columns of $A^t$ are the rows of $A$ and $A$ and $A^t$ have the same rank, so the rank of $A$ is the dimension of its row space.

(c) By part (b) and Theorem 3.5, we have that the dimension of the column space is the same as the dimension of the row space.

\[\square\]

**Cor 3.** Every invertible matrix is a product of elementary matrices.

**Proof.** Let $A \in M_n(F)$, $A$ invertible. By Fact 8, the range of $A$ is $F^n$ and so has rank $n$. By Theorem 3.6 $I_n = BAC$ for where $B$ and $C$ are both products of elementary matrices. As a result, $B$ and $C$ are both invertible and their inverses are products of elementary matrices. $(E_1E_2)^{-1} = E_2^{-1}E_1^{-1}$. We have that $A = B^{-1}I_nC^{-1} = B^{-1}C^{-1}$ which is a product of elementary matrices.

\[\square\]

**Chapter 4**

**Defn 4.** (p. 209) Let $A \in M_{n \times n}(F)$. If $n = 1$, we define $\det(A) = A_{1,1}$. For $n \geq 2$, we define $\det(A)$ recursively as

$$\det(A) = \sum_{j=1}^{n} (-1)^{1+j} A_{1,j} \cdot \det(\tilde{A}_{1,j}).$$

The scalar $\det(A)$ is called the **determinant** of $A$ and is also denoted by $|A|$. The scalar $(-1)^{i+j} |\tilde{A}_{i,j}|$ is called the **cofactor** of the entry of $A$ in row $i$, column $j$. This formula for the determinant is called the **cofactor expansion along the first row** of $A$.

**Theorem 4.3.** The determinant of an $n \times n$ matrix is a linear function of each row when the remaining rows are held fixed.

**Cor 1.** If $A \in M_n(F)$ has a row consisting entirely of zeros, then $\det(A) = 0$.

**Theorem 4.4.** The determinant of a square matrix can be evaluated by cofactor expansion along any row.

**Cor 1.** If $A \in M_n(F)$ has two identical rows, then $\det(A) = 0$.

**Theorem 4.5.** If $A \in M_n(F)$ and $B$ is a matrix obtained from $A$ by interchanging any two rows of $A$, then $\det(B) = -\det(A)$.

**Theorem 4.6.** Let $A \in M_n(F)$, and let $B$ be a matrix obtained by adding a multiple of one row of $A$ to another row of $A$. Then $\det(B) = \det(A)$.
Cor 1. If \( A \in M_n(F) \) has rank less than \( n \), then \( \det(A) = 0 \).

Fact 12. (p. 223)

(a) If \( E \) is an elementary matrix obtained by interchanging any two rows of \( I \), then \( \det(E) = -1 \).

(b) If \( E \) is an elementary matrix obtained by multiplying some row of \( I \) by the nonzero scalar \( k \), then \( \det(E) = k \).

(c) If \( E \) is an elementary matrix obtained by adding a multiple of some row of \( I \) to another row, then \( \det(E) = 1 \).

Theorem 4.7. For any \( A, B \in M_n(F) \), \( \det(AB) = \det(A) \cdot \det(B) \).

Cor 1. A matrix \( A \in M_n(F) \) is invertible if and only if \( \det(A) \neq 0 \). Furthermore, if \( A \) is invertible, then \( \det(A^{-1}) = \frac{1}{\det(A)} \).

Theorem 4.8. For any matrix \( A \in M_n(F) \), \( \det(A^t) = \det(A) \).