Introduction to Mathematical Rigor, MTH 307

Exercises

1.45 F3 If \( x \leq y \) and \( u \leq v \) then \( x + u \leq y + v \)

If \( x = y \) then by F1, \( u \leq v \) implies \( u + x \leq v + x \). But \( x = y \), so \( v + x = v + y \). So, \( u + x \leq v + y \) and we have \( x + u \leq y + v \) by commutativity.

If \( u = y \) then similarly we have \( x + u \leq y + v \).

So assume \( x < y \) and \( u < v \). That is, \( y - x \in P \) and \( v - u \in P \). Thus, by P1, \( (y - x) + (v - u) \in P \). But,

\[
(y - x) + (v - u) = y + (-x) + v + (-u) = y + (-1)x + v + (-1)u = y + v + (-1)x + (-1)u = (y + v) + (-1)(x + u) = (y + v) - (x + u)
\]

So, \( (y + v) - (x + u) \in P \) which means, \( x + u \leq y + v \).

1.45, F4 If \( 0 \leq x \leq y \) and \( 0 \leq u \leq v \) then \( xu \leq yv \)

We have \( x \leq y \) and \( 0 \leq u \). By F2, we have \( xu \leq yu \). Also, we have \( u \leq v \) and \( 0 \leq y \).

So, by F2 again, we have \( uy \leq vy \) which implies by Commutativity that \( yu \leq vy \).

Combining \( xu \leq yu \) and \( yu \leq vy \), we have by O3, \( xu \leq yv \).

1.46, b \( x \leq y \) and \( z \leq 0 \) implies \( yz \leq xz \).

If \( z = 0 \) then \( xz = 0 \) and \( yz = 0 \), so \( yz = xz \).

Assume \( z < 0 \). So \( -z \in P \).

If \( x = y \) then \( xz = yz \) so \( yz \leq xz \).

The only remaining case to consider is when \( x < y \). Then \( y - x \in P \). So, \( (y - x)(-z) \in P \). But \( (y - x)(-z) = y(-z) + (-x)(-z) \). By 1.43e, we have \( (-x)(-z) = xz \) and by 1.43c, we have \( (-z)y = -(zy) \). So \( -zy + xz \in P \). That is, \( xz - yz \in P \). So, \( yz \leq xz \).

1.46, e \( 0 < 1 \).

By Trichotomy, either \( 1 - 0, 1 \in P \), or \( -1 \in P \).

If \( 1 = 0 \) then \( x \cdot 1 = x \) implies \( x \cdot 0 = x \). Which of course is not true for \( x \neq 0 \).

We know \( 1 \cdot 1 = 1 \). So \( 1^2 = 1 \). By (d), \( 1 \geq 0 \). We know \( 1 \neq 0 \), so \( 1 > 0 \).

1.46, f \( 0 < x \) implies \( 0 < x^{-1} \)

Suppose \( x^{-1} \leq 0 \) Then by 1.46b, \( 0 \leq x \) implies \( x \cdot x^{-1} \leq 0 \cdot x^{-1} \). So \( 1 \leq 0 \). This is not true by (e), so we have reached a contradiction. That means that \( x^{-1} > 0 \).
1.46, $g \ 0 < x < y$ implies $0 < y^{-1} < x^{-1}$.

We have $0 < y^{-1}$ and $0 < x^{-1}$, from (f).

So $x < y$ and $0 < x^{-1}$ implies $xx^{-1} < yy^{-1}$, by F2.

So $1 < y \cdot x^{-1}$

$y^{-1} < y^{-1}yx^{-1}$

$y^{-1} < 1 \cdot x^{-1}$

$y^{-1} < x^{-1}$