You can improve your grade on Exam 1 by completing this assignment and handing it in by October 31. Your grade on this will be averaged with the grade on Exam 1 to give you a replacement grade for Exam 1. **Do not get help from anyone but me on this.**

(1) (20pts) Let \( f \) and \( g \) be functions, from \( \mathbb{R} \) to \( \mathbb{R} \), where \( f(x) = 3x + 1 \) and \( g(x) = x^2 - 4 \).

(a) Let \( A \) be the image of \( f \). Describe \( A \) as a subset of \( \mathbb{R} \).

(b) Let \( B \) be the image of \( g \). Describe \( B \) as a subset of \( \mathbb{R} \).

(c) If \( y = b^2 - 4 \), find \( a \) such that \( y = 3a + 1 \).

(d) Prove or disprove \( A \subseteq B \),

(e) Prove or disprove \( B \subseteq A \).

(2) (20pts) A subset \( S \) of \( \mathbb{R} \) is bounded if there exists an \( M \in \mathbb{R} \) such that \( |x| \leq M \) for all \( x \) in \( S \). Let \( T = \{2k + 1 : k \in \mathbb{Z}\} \).

(a) Let \( x \in T \), is \( x \in \mathbb{Z} \)? Why or why not?

(b) Let \( x \in T \), is \( x \in \mathbb{R} \)? Why or why not?

(c) Rewrite the following statement using quantifiers and without using the words bounded or unbounded:

\[ T \text{ is bounded.} \]

(d) Negate the above statement.

(e) Circle the true statements: (c) (d).
(3) (24pts) Let $F$ be an ordered field and let $x, y, z \in F$.

(a) Justify each step of the following proof that $0 \cdot x = 0$ with a field axiom.

\[
\begin{align*}
1 \cdot x &= x \\
(1 + 0)x &= x \\
1 \cdot x + 0 \cdot x &= x \\
x + 0 \cdot x &= x \\
(−x) + (x + 0 \cdot x) &= −x + x \\
(−x + x) + 0 \cdot x &= 0 \\
0 + 0 \cdot x &= 0 \\
0 \cdot x &= 0
\end{align*}
\]

(b) Prove that $(−x) \cdot y = −(xy)$. Only use field axioms and fact (a).

(c) Using only axioms for an ordered field and facts (a), (b), and $(−x)(−y) = xy$, prove that $x < y$ and $z < 0$ implies $xz > yz$. 
(4) (16pts) Justify each step of the following proof of the AGM inequality with one or more properties of ordered fields given in Propositions 1.42 through 1.46.

(a) 
\[
\begin{align*}
0 & \leq (a - b)^2 \\
2ab & \leq a^2 + b^2 \\
4ab & \leq (a + b)^2 \\
ab & \leq \frac{(a + b)^2}{4}
\end{align*}
\]

(b) Substitute \(a = xv\) and \(b = yu\) into \(2ab \leq a^2 + b^2\)

(c) Starting with (b), show that \((xu + yv)^2 \leq (x^2 + y^2)(u^2 + v^2)\).

(5) (20pts) Consider the statement \(P\): If \(x \geq -1\) then \(|x + 1| = |x| + 1\).

(a) Form the contrapositive.

(b) Form the converse.

(c) Form the negation.

(d) Use the definition of the absolute value to show that if \(x \geq 1\) then \(|x - 1| = x - 1\) and \(|x| = x\).

(e) Prove or disprove \(P\).