Theorem 1.10 (Replacement Theorem). Let \( V \) be a vector space generated by a set \( G \) containing exactly \( n \) vectors, and let \( L \) be a linearly independent subset of \( V \) containing exactly \( m \) vectors. Then \( m \leq n \) and there exists a subset \( H \) of \( G \) containing exactly \( n - m \) vectors such that \( L \cup H \) generates \( V \).

Proof. The proof is by mathematical induction on \( m \). The induction begins with \( m = 0 \); for in this case \( L = \emptyset \), and so taking \( H = G \) gives the desired result.

Now suppose that the theorem is true for some integer \( m \geq 0 \). We prove that the theorem is true for \( m + 1 \).

Suppose \( V \) is a vector space that is generated by a set \( G \) containing exactly \( n \) vectors, and \( L = \{v_1, \ldots, v_m\} \) is a linearly independent subset of \( V \) consisting of \( m + 1 \) vectors.

Moreover, some \( b_i \), say \( b_1 \) is nonzero, for otherwise we obtain the same contradiction. Solving (1) for \( u_1 \) gives
\[
\begin{align*}
  u_1 &= \sum_{i=1}^{m} (-b_i^{-1}b_m) v_i + \sum_{i=m+1}^{n} (-b_i^{-1}b_{m+1}) v_i + \cdots + (-b_i^{-1}b_{n-m}) v_{n-m} \\

  &= \sum_{i=1}^{m} (-b_i^{-1}b_m) v_i + \sum_{i=m+1}^{n} (-b_i^{-1}b_{m+1}) v_i + \cdots + (-b_i^{-1}b_{n-m}) v_{n-m}.
\end{align*}
\]

Let \( H = \{u_2, \ldots, u_{n-m}\} \). Then \( u_1 \in \text{span}(L \cup H) \), and because \( L' \cup H \subseteq \text{span}(L \cup H) \), it follows that \( L' \cup H' \subseteq \text{span}(L \cup H) \). We have \( \text{span}(L' \cup H') = V \) by the induction hypothesis.

Theorem 1.5 implies that \( \text{span}(L' \cup H') \subseteq \text{span}(L \cup H) \). So \( V \subseteq \text{span}(L \cup H) \) so \( V = \text{span}(L \cup H) \).

Since \( H \) is a subset of \( G \) that contains \( (n - m) - 1 = n - (m + 1) \) vectors, the theorem is true for \( m + 1 \). This completes the induction.

By the corollary to Theorem 1.6, \( L' = \{v_1, \ldots, v_m\} \) is linearly independent, and so we may apply the induction hypothesis to conclude that \( m \leq n \) and that there is a subset \( H' = \{u_1, \ldots, u_{n-m}\} \) of \( G \) such that \( L' \cup H' \) generates \( V \).

Thus there exist scalars \( a_1, \ldots, a_m, b_1, \ldots, b_{n-m} \) such that
\[
a_1 v_1 + \cdots + a_m v_m + b_1 u_1 + \cdots + b_{n-m} u_{n-m} = v_{m+1}.
\]

Note that \( n - m > 0 \), lest \( v_{m+1} \) be a linear combination of \( v_1, \ldots, v_m \), which by Theorem 1.7, contradicts the assumption that \( L \) is linearly independent. Hence \( n > m \); that is, \( n \geq m + 1 \), which is one of the conclusions we needed to make.

Cor 3 (1, To Theorem 1.10) Let \( V \) be a vector space having a finite basis. Then every basis for \( V \) contains the same number of vectors.

Proof. Let \( \beta \) be a finite basis of \( V \); let \( |\beta| = n \). Let \( \gamma \) be another basis of \( V \). Suppose \( \gamma \) contains \( n + 1 \) (or more) elements. Select subset \( S \subseteq \gamma \) such that \( |S| = n + 1 \). Then \( S \) is linearly independent. If we apply the Replacement Theorem with \( G = \beta \) and \( S = L \), we obtain \( n + 1 \leq n \).

This is a contradiction, so \( |\gamma| \leq |\beta| \).

Then reversing roles of \( \gamma \) and \( \beta \), we get
\[
|\beta| \leq |\gamma|.
\]
**Defn 8** A vector space $V$ is called **finite-dimensional** if it has a finite basis. The size of which is called the dimension of $V$, denoted $\dim V$. A vector space that is not finite-dimensional is called **infinite-dimensional**.

**Exercise 15** (Section 1.5) Let $S = \{u_1, u_2, \ldots, u_n\}$ be a finite set of vectors. Prove that $S$ is linearly dependent if and only if $u_1 = 0$ or $u_{k+1} \in \text{span}(\{u_1, u_2, \ldots, u_k\})$ for some $k$ ($1 \leq k \leq n$).

**Cor 4** (2, To Theorem 1.10) Let $V$ be a vector space with dimension $n$.

1. Any finite generating set of $V$ contains at least $n$ vectors, and a generating set for $V$ that contains exactly $n$ vectors is a basis for $V$.

**Proof.** Suppose $S$ generates $V$ and $|S| = m$. If $S$ is a basis, then $m = n$. If not, it must be linearly dependent. Then some vector $v \in S$ can be written as a linear combination of the other vectors in $S$. But $\text{span}(S - v) = \text{span}(S)$. We can remove vectors from $S$ until we find a subset $S' \subseteq S$, such that $S'$ is linearly independent and $\text{span}(S') = \text{span}(S)$. We know $|S'| = n$, therefore, $|S| = m \geq n$. □

2. Any linearly independent subset of $V$ that contains exactly $n$ vectors is a basis for $V$.

**Proof.** Let $L$ be a linearly independent subset of $V$ with exactly $n$ vectors. We know there is a set $G$ of $n$ vectors that generates $V$, since $\dim V = n$. By the Replacement Theorem there is a set $H$ of $n - n = 0$ vectors such that $L \cup H = L$ generates $V$. Thus $L$ generates $V$ and is a basis. □
3. Every linearly independent subset of $V$ can be extended to a basis for $V$.

**Proof.** Let $G$ be a basis for $V$ of $n$ vectors and $L$ a linearly independent subset of $V$. By the Replacement Theorem there is a subset $H$ of $V$ such that $L \cup H$ generates $V$ and $|L \cup H| = n$. So by (1.), $L \cup H$ is a basis for $V$. $\square$

**Theorem 1.11** Let $W$ be a subspace of a finite-dimensional vector space $V$. Then $W$ is finite-dimensional and $\dim(W) \leq \dim(V)$. Moreover, if $\dim(W) = \dim(V)$, then $V = W$.

**Proof.** If $W = \{0\}$ then $\dim(W) = 0$ and we have the result.

If there is a subset $L \subseteq W$, such that $L$ is linearly independent. We know that $V$ has a basis $G$ of $n$ vectors that spans $V$. By the replacement $|L| \leq n$.

We define a linearly independent subset of $W$ inductively. Assume there is a non-zero $x_1 \in W$. Then $L_1 = \{x_1\}$ is a linearly independent set. Let $i > 1$ and assume $L_{i-1}$ has $i - 1$ linearly independent vectors from $W$ in it. If there is a vector $x_i$ in $W$ such that $L_{i-1} \cup \{x_i\}$ is linearly independent, we set $L_i = L_{i-1} \cup \{x_i\}$. If not, we set $L = L_{i-1}$. So that $L_i = \{x_1, x_2, \ldots, x_k\}$ is a linearly independent set and by Corollary 1.7, it spans $W$.

We know that $L$ is finite and the process ends be the reasoning above. And we have that $|L| = k = \dim W$. If $m = n$, then by Corollary 2(2.), $L$ must be a basis of $V$. Thus, $W = V$. $\square$

**Corollary 1** (to Theorem 1.11) If $W$ is a subspace of a finite-dimensional vector space $V$, then any basis for $W$ can be extended to a basis for $V$.

**Proof.** Let $L$ be a basis of $W$ and $G$ be a basis of $V$. By Theorem 1.11, $|L| \leq |G|$. By Corollary 2(c), $L$ can be extended to a basis of $V$. $\square$