Let \( z = f(x, y) \) be given. The partial derivatives \( f_x(a, b), f_y(a, b) \) at some point \((a, b)\) give the rates of change in the direction of \(x\) and the direction of \(y\):

\[
\begin{align*}
\nabla f(a, b) &= \begin{pmatrix} f_x(a, b) \\ f_y(a, b) \end{pmatrix}
\end{align*}
\]

\(fx(a, b)\) gives us the rate of change when we walk from \((a, b)\) in the direction of \(\hat{i}\), \(f_y(a, b)\) if we walk in the direction of \(\hat{j}\).

What is the rate of change if we move from \((a, b)\) in the direction of some other unit vector \(\vec{u}\)?

This rate of change is called the directional derivative of \(f(x, y)\) at \((a, b)\) in the direction of \(\vec{u}\).

**Def:** Let \(\vec{u} = u_1 \hat{i} + u_2 \hat{j}\), \(\|\vec{u}\| = 1\), \(z = f(x, y)\), \((a, b)\) be given. The directional derivative of \(f(x, y)\) at \((a, b)\) in the direction of \(\vec{u}\) is:

\[
f_{\vec{u}}(a, b) = \lim_{h \to 0} \frac{f(a + hu_1, b + hu_2) - f(a, b)}{h}
\]
Note: The definition talks about both-sided limit "\( \lim \)". Thus \( h \) can be positive or negative.

The limit exists if the cross-section of the graph by the vertical plane containing \( \overrightarrow{u} \) and \((a, b, f(a,b))\) has the tangent line at \((a, b, f(a,b))\):

Thus \( f_{\overrightarrow{u}}(a,b) \) also tells us that if we go in the opposite direction, \(-\overrightarrow{u}\), \(f(x,y)\) will be changing at the rate \( -f_{\overrightarrow{u}}(a,b) = f_{-\overrightarrow{u}}(a,b) \)

Indeed:

\[
f_{-\overrightarrow{u}}(a,b) = \lim_{h \to 0} \frac{f(a + h(-u_1), b + h(-u_2)) - f(a,b)}{h}
\]

\[
= \lim_{h \to 0} \frac{f(a + (h)u_1, b + (h)u_2) - f(a,b)}{-h}
\]

\[
= - f_{\overrightarrow{u}}(a,b)
\]
Note, the displacement vector from \((a, b)\) to \((a+h u_1, b+h u_2)\) is 
\[ \bar{d} = (h u_1) \bar{i} + (h u_2) \bar{j} = \bar{h} \cdot \bar{u}, \]
so \(\|\bar{d}\| = |h|\) (or more precisely \(\|h\|\)).

This is why we need \(\bar{u}\) to be the unit vector so the magnitude of \(h\) is the magnitude of the displacement. Geometrically, \(f_{\bar{u}} (a, b)\) is the slope of the cross-section with the vertical plane parallel to \(\bar{u}\), through \((a, b, f(a, b))\). Clearly:

\[ f_x (a, b) = f_{\bar{i}} (a, b), \quad f_y (a, b) = f_{\bar{j}} (a, b). \]

**Def:** Let \(f(x, y), (a, b),\) and a vector \(\bar{v}\) be given. Then

\[ f_{\bar{v}} (a, b) = \frac{f(\bar{v})}{\|\bar{v}\|} (a, b). \]

How do we calculate directional derivatives? Using the so-called gradient vector.

**Def:** The gradient vector of \(f(x, y)\) at \((a, b)\) is defined as

\[ \text{grad} f (a, b) = f_x (a, b) \bar{i} + f_y (a, b) \bar{j}. \]
In general:
\[ \nabla f(x, y) = f_x(x, y) \mathbf{i} + f_y(x, y) \mathbf{j} \]
or
\[ \nabla f = f_x \mathbf{i} + f_y \mathbf{j} \]

Ex: Let \( f(x, y) = x^3 y + 3y + x \). Find \( \nabla f(x, y) \). Find \( \nabla f(1, 0) \).

\( f_x = 3x^2 y + 1, \quad f_y = x^3 + 3 \)

\[ \nabla f = (3x^2 y + 1) \mathbf{i} + (x^3 + 3) \mathbf{j} \]

\[ \nabla f(1, 0) = \mathbf{i} + 4 \mathbf{j} \]

\( \nabla f \) is a vector!

Th: Let \( f(x, y) \) be differentiable at \((a, b)\), \( \mathbf{u} \) be the unit vector. Then

\[ f_{\mathbf{u}}(a, b) = \nabla f(a, b) \cdot \mathbf{u} =
\]

\[ = f_x(a, b) u_1 + f_y(a, b) u_2 \]

Gradient is also denoted:
\[ \nabla f = \nabla f \]
Ex: Let \( \mathbf{u} = \frac{1}{\sqrt{2}} \mathbf{i} + \frac{1}{\sqrt{2}} \mathbf{j} \), \( f(x, y) = x^2 + y^2 \).

Find \( \mathbf{f}_{\mathbf{u}} (1, 0) \).

We use gradient.

\[
\text{grad } f(x, y) = 2x \mathbf{i} + 2y \mathbf{j}
\]

\[
\text{grad } f(1, 0) = 2 \mathbf{i}
\]

\[
\mathbf{f}_{\mathbf{u}} (1, 0) = 2 \mathbf{i} \cdot \mathbf{u} = 2 \cdot \frac{1}{\sqrt{2}} = \sqrt{2}.
\]

So finding directional derivatives is as easy as finding gradients.

Let \( f(x, y) \), differentiable, be given. Then we have the gradient vector at each point:

\[
\nabla f(x, y)
\]

So we have a vector field - the gradient field on the \( xy \)-plane:

\[
\nabla f(x, y)
\]
Gradient Geometrically

Let's have \( f(x, y) \), \( (a, b) \), \( \nabla f(a, b) \):

\[
\begin{align*}
\vec{u} & \rightarrow \nabla f(a, b) \quad \text{How about directional derivatives at} \ (a, b) \ ? \\
\text{Let} \ \vec{u} \ \text{be a unit vector.}
\end{align*}
\]

\[
f_{\vec{u}}(a, b) = \nabla f(a, b) \cdot \vec{u} = \| \nabla f(a, b) \| \| \vec{u} \| \cos(\Theta)
\]

\( f_{\vec{u}}(a, b) \) is the largest if \( \cos(\Theta) = 1 \); that is, when \( \Theta = 0 \), so \( \vec{u} \) is parallel to \( \nabla f(a, b) \).

Thus:

**Th:** Let \( f(x, y) \) be differentiable at \( (a, b) \), \( \nabla f(a, b) \neq \vec{0} \).

Then:

1. \( \nabla f(x, y) \) points in the direction of the largest rate of change of \( f \) at \( (a, b) \). (The direction of the fastest growth.)

2. This largest rate of change is \( \| \nabla f(a, b) \| \).

3. \( \nabla f(a, b) \) is perpendicular to the contour line of \( f(x, y) \) through \( (a, b) \).
Why (3)? Since \( f(x,y) \) is differentiable at \((a,b)\), locally near \((a,b)\) the graph of \( z = f(x,y) \) is flat. Thus, locally contour lines are parallel:

You obtain fastest growth by moving perpendicularly to contours in the direction of increasing values. So \( \nabla f(a,b) \) that points toward fastest increase must be perpendicular to contours.

\textbf{Ex:} Let \( f(x,y) = x^2 + y^2 \). Sketch the contour diagram and the gradient field in one coordinate system.

\[ \nabla f(x,y) = 2xi + 2yj \]

Parallel the vector from \((0,0)\) to \((x,y)\).

The path of steepest ascent.

Magnitudes of vectors are scaled.
Ex: A square metal plate is placed in the xy-plane in such a way that \(0 \leq x \leq 3\), \(0 \leq y \leq 3\), [All measurements in meters.] The temperature at each point \((x, y)\) of the plane is given by:

\[
T(x, y) = \frac{100}{x^2 + y^2 + 1} \text{ in } ^\circ F.
\]

(a) Find the direction of the greatest increase in temp. at \((1, 2)\). What is the greatest rate of increase at the point \((1, 2)\)?

(b) Find the rate of increase at \((1, 2)\) in the direction \(\overrightarrow{u} = \frac{2}{\sqrt{5}} \overrightarrow{i} + \frac{1}{\sqrt{5}} \overrightarrow{j}\).

Clearly, we have to find the gradient \(\nabla T(x, y)\) first.

\[
\nabla T(x, y) = \frac{\partial}{\partial x} \left[ \frac{100}{x^2 + y^2 + 1} \right] \hat{i} + \frac{\partial}{\partial y} \left[ \frac{100}{x^2 + y^2 + 1} \right] \hat{j} =
\]

\[
= 100 \cdot -\frac{2x}{(x^2 + y^2 + 1)^2} \hat{i} = -\frac{200x}{(x^2 + y^2 + 1)^2} \hat{i}.
\]

\[
T_x (x, y) = -\frac{100}{(x^2 + y^2 + 1)^2} \cdot 2x = -\frac{200x}{(x^2 + y^2 + 1)^2}.
\]

\[
T_y (x, y) = -\frac{100}{(x^2 + y^2 + 1)^2} \cdot 2y
\]

\[
\nabla T(x, y) = \left( -\frac{100}{(x^2 + y^2 + 1)^2} \right) \left( 2x \hat{i} + 2y \hat{j} \right).
\]

\[
\nabla T(1, 2) = \left( -\frac{100}{(1^2 + 2^2 + 1)^2} \right) \left( 2 \hat{i} + 2 \hat{j} \right).
\]

\[
\nabla T(1, 2) = \left( -\frac{100}{(6)^2} \right) \left( 2 \hat{i} + 2 \hat{j} \right).
\]

\[
\nabla T(1, 2) = \left( -\frac{100}{36} \right) \left( 2 \hat{i} + 2 \hat{j} \right).
\]

\[
\nabla T(1, 2) = \left( -\frac{25}{9} \right) \left( 2 \hat{i} + 2 \hat{j} \right).
\]

\[
\nabla T(1, 2) = \left( -\frac{50}{9} \hat{i} - \frac{50}{9} \hat{j} \right).
\]
Contours are concentric circles centered at $\mathbf{0}$. At each point the gradient $\nabla T(x,y)$ points toward the origin.

$$\nabla T(1,2) = -\frac{200}{36} \mathbf{i} - \frac{400}{36} \mathbf{j}$$

The direction of the greatest increase in temperature at $(1,2)$ (toward the origin). This answers (a).

(b) $\lVert \nabla T(1,2) \rVert = \sqrt{\left(\frac{200}{36}\right)^2 + \left(\frac{400}{36}\right)^2} \approx 77.16 \, ^\circ F$ per meter

(C) $\frac{T}{u^2}(1,2) = \nabla T(1,2) \cdot \mathbf{u}^2 =$

$$= -\frac{200}{36} \cdot \frac{2}{15} - \frac{400}{36} \cdot \frac{1}{15} \approx -9.93 \, ^\circ F$$
14.5 Gradients of Functions of Three Variables

\[ f(x, y, z) = x^2 + y^2 + z^2 \]

We cannot "graph" functions of three variables. We can graph their level surface for a given \( k \):
\[ x^2 + y^2 + z^2 = k. \]

We can, of course, define three partial derivatives at each point \((a, b, c)\):
\[ f_x(a, b, c), \quad f_y(a, b, c), \quad f_z(a, b, c) \]
by fixing two variables and differentiating with respect to the third. For example:
\[ f_z(a, b, c) = \frac{d}{dz} \bigg|_{z=c} \left[ f(a, b, z) \right]. \]

In our example:
\[ f_x(x, y, z) = 2x, \quad f_y(x, y, z) = 2y, \quad f_z(x, y, z) = 2z. \]

We can define the directional derivative of a function of three variables \( f(x, y, z) \) in the direction of \( \mathbf{u} = u_1 \mathbf{i} + u_2 \mathbf{j} + u_3 \mathbf{k} \), with \( \| \mathbf{u} \| = 1 \):
\[ f_u (a, b, c) = \lim_\limits{h \to 0} \frac{f(a+hu_1, b+hu_2, c+hu_3) - f(a, b, c)}{h} \]

This is the rate of change in the direction of \( \vec{u} \).

We can define the gradient vector:

\[ \nabla f(x, y, z) = f_x(x, y, z) \vec{i} + f_y(x, y, z) \vec{j} + f_z(x, y, z) \vec{k} \]

In our example, \( f(x, y, z) = x^2 + y^2 + z^2 \)

\[ \nabla f = 2x \vec{i} + 2y \vec{j} + 2z \vec{k} \]

As before, if \( f(x, y, z) \) is differentiable at \( (a, b, c) \):

\[ f_u (a, b, c) = \nabla f(a, b, c) \cdot \vec{u} = \frac{\text{units of } f}{\text{units of } \text{dist in dir } \vec{u}} \]

As before:

If \( \nabla f(a, b, c) \neq \vec{0} \), then

- \( \nabla f(a, b, c) \) points in the direction of the greatest rate of change of \( f \) at \( (a, b, c) \).
- \( \| \nabla f(a, b, c) \| \) is the greatest rate of change
- \( \nabla f(a, b, c) \) is perpendicular to the level surface through \( (a, b, c) \).
Ex. Suppose that the function \( F(x, y, z) = x^2 + y^4 + x^2 z^2 \) gives concentration of salt, in gr/gal, at any point \((x, y, z)\) of a rectangular tank of water occupying the region

\[-2 \leq x \leq 2, \ -2 \leq y \leq 2, \ 0 \leq z \leq 2,\]

(All measurements in meters.) Suppose you are at the point \((-1, 1, 1)\).

(a) In what direction should you move if you want the concentration to increase the fastest?

(b) If you move from \((-1, 1, 1)\) toward the origin \((0,0,0)\), how fast is the concentration changing?

\[\nabla F(x, y, z) = (2x + 2xz^2)i + 4y^3j + 2x^2z k\]

\[\nabla F(-1, 1, 1) = -4i + 4j + 2k\]

The direction of greatest increase in concentration.

\[\| \nabla F(-1, 1, 1) \|= \sqrt{16 + 16 + 4} = 6 \frac{g}{gal m}\]

\[\vec{v} = (-1, 1, 1)(0,0,0) = i - j - k, \quad \| \vec{v} \| = \sqrt{3}\]
\[ \vec{w} = \frac{1}{\sqrt{3}} \vec{i} - \frac{1}{\sqrt{3}} \vec{j} - \frac{1}{\sqrt{3}} \vec{k} \]

\[ F_{\vec{w}} (-1, 1, 1) = F_{\vec{w}} (-1, 1, 1) = \nabla F (-1, 1, 1) \cdot \vec{w} = \]

\[ = - \frac{4}{\sqrt{3}} - \frac{4}{\sqrt{3}} - \frac{2}{\sqrt{3}} = - \frac{10}{\sqrt{3}} \approx -5.77 \frac{8}{9} \mathrm{m} \]

**Example:** Find the equation of the tangent plane to the ellipsoid \( x^2 + 2y^2 + z^2 = 15 \) at \((2, 1, 3)\).

The ellipsoid is the level surface of \( F(x, y, z) = x^2 + 2y^2 + z^2 \)

\[ \vec{n} = \nabla F (2, 1, 3) \]

\[ \nabla F = 2x \vec{i} + 4y \vec{j} + 2z \vec{k} \]

\[ \vec{n} = \nabla F (2, 1, 3) = 4 \vec{i} + 4 \vec{j} + 6 \vec{k} \]

\( P = (2, 1, 3) \)

Plane: \( 4(x-2) + 4(y-1) + 6(z-3) = 0 \)