You are familiar with parametric representations of curves on the \(xy\)-plane.

**Ex:** What curve on the \(xy\)-plane is described by:

\[
x = \cos t, \quad y = \sin t, \quad t \in [0, 2\pi]
\]

Since \(\cos^2 t + \sin^2 t = 1\), each point on the parametric curve is on the unit circle:

\[
t = 0 \rightarrow (1, 0)
\]

As \(t\) increases, we move counterclockwise; at \(t = 2\pi\), we are back at \((1, 0)\).

Of course, the unit circle has other parametric representations. For example:

\[
x = \sin t, \quad y = \cos t, \quad t \in [0, 2\pi]
\]

Same path, different parametrization.

Parametrizations provide a very convenient way of representing curves in the \(xyz\)-space.
A parametric curve in the \(xyz\)-space is a curve described by parametric equations:

\[ x = f(t), \quad y = g(t), \quad z = h(t) \]

where the parameter \(t\) changes in an interval \(I\).

For each \(t\) we have a point \((f(t), g(t), h(t))\) on the curve \(C\). As \(t\) changes, the point moves along \(C\) describing a motion along the path \(C\).

**Example**: What curve is described by

\[ x = \cos t, \quad y = \sin t, \quad z = t, \quad t \geq 0. \]

What motion along the curve is described by the parametrization?

Note that the projection onto the \(xy\)-plane:

\[ x = \cos t, \quad y = \sin t \]

moves \(ccw\) about the unit circle. At the same time \(z\) increases so we move up.

A helix in the \(xyz\)-space.
The same helix can be parametrized by:

\[
\mathbf{r}(t) = \cos(2t) \hat{i} + \sin(2t) \hat{j} + t \hat{k}, \quad t \geq 0.
\]

In the latter parametrization we traverse the helix twice as fast.

---

**Parametrization in Vector Form**

Let a parametrized curve

\[
x = f(t), \quad y = g(t), \quad z = h(t), \quad t \in I
\]

be given. We can write the parametrization as:

\[
\mathbf{r}(t) = f(t) \hat{i} + g(t) \hat{j} + h(t) \hat{k}, \quad t \in I.
\]

\(\mathbf{r}(t)\) is called the **position vector**:

The position vector traces the curve as \(t\) changes.

The simplest way to describe a straight line in 3D is by a parametric representation.
Let \( L \) be the line passing through a point \((x_0, y_0, z_0)\) and parallel to \( \mathbf{w} = w_1 \mathbf{i} + w_2 \mathbf{j} + w_3 \mathbf{k} \). Then
\[
L : x(t) = x_0 + tw_1, \quad y(t) = y_0 + tw_2, \quad z(t) = z_0 + tw_3, \quad -\infty < t < +\infty.
\]

In the vector form:
\[
L : \mathbf{r}(t) = \mathbf{r}_0 + t \mathbf{w}, \quad \mathbf{r}_0 = x_0 \mathbf{i} + y_0 \mathbf{j} + z_0 \mathbf{k}, \quad -\infty < t < +\infty
\]

Of course, the displacement vector from \((x_0, y_0, z_0)\) to \((x(t), y(t), z(t))\) is \( tw_1 \mathbf{i} + tw_2 \mathbf{j} + tw_3 \mathbf{k} \parallel \mathbf{w} \). So we start from \((x_0, y_0, z_0)\) and move \( t \) units in the direction of \( \mathbf{w} \).

\[
(0, 0, 0) \quad \mathbf{r}_0 \quad \mathbf{r}_0 + t \mathbf{w} \quad (x(t), y(t), z(t))
\]

\( \mathbf{r}_0 \) is the position vector of the initial point.

**Ex:** Let \( P_0 = (2, -1, 3) \), \( P_1 = (-1, 5, 4) \). Find a parametric representation of:

(a) The line, \( L_1 \), through \( P_0, P_1 \)

(b) The segment, \( S \), from \( P_0 \) to \( P_1 \).
(a) \( \vec{w} = (2, -1, 3) \cdot (-1, 5, 4) = -3\vec{i} + 6\vec{j} + \vec{k} \) - the direction vector

\( P_0 = (x_0, y_0, z_0) = (2, -1, 3) \) - initial point

\( \vec{r}_0 = 2\vec{i} - \vec{j} + 3\vec{k} \) - the position vector of the initial point.

Parametric representation:

\[ L: \vec{r}(t) = \vec{r}_0 + t\vec{w}, \quad -\infty < t < +\infty \]

Thus:

\[ L: \vec{r}(t) = 2\vec{i} - \vec{j} + 3\vec{k} + t(-3)\vec{i} + t6\vec{j} + t\vec{k} \]

\[ L: \vec{r}(t) = (2-3t)\vec{i} + (-1+6t)\vec{j} + (3+t)\vec{k} \]

In non-vector form:

\[ L: \quad x(t) = 2-3t, \quad y(t) = -1+6t, \quad z(t) = 3+t \]

\[ -\infty < t < +\infty \]

(b) Since \( \vec{w} = \vec{P_0P_1} \), in the representation

\[ \vec{r}(t) = \vec{r}_0 + t\vec{w} \]

we are at \( P_0 \) for \( t=0 \) and at \( P_1 \) at \( t=1 \).

So:

\[ S: \quad \vec{r}(t) = (2-3t)\vec{i} + (-1+6t)\vec{j} + (3+t)\vec{k} \]

\[ 0 \leq t \leq 1 \]

Note: A parametric representation gives us not only a path (L or S) but also a specific motion along the path.
Ex: Find a parametric equation of the circle of radius 3 parallel to the xy-plane centered at (0, 0, 2).

\[ x = 3 \cos t, \ y = 3 \sin t, \ z = 2 \]

\[ 0 \leq t \leq 2\pi \]

Or

\[ \vec{r}(t) = (3 \cos t) \hat{i} + (3 \sin t) \hat{j} + 2 \hat{k} \]

\[ 0 \leq t \leq 2\pi. \]

Of course, we use parametric representations to describe motion in xyz-space. The parameter \( t \) denotes time. What is the velocity and the acceleration?

**Def:** Let \( \vec{r}(t) = f(t) \hat{i} + g(t) \hat{j} + h(t) \hat{k} \) be the position vector of an object at time \( t \). The velocity vector \( \vec{V}(t) \), at time \( t \) is defined as:

\[ \vec{V}(t) = \lim_{\Delta t \to 0} \frac{\vec{r}(t+\Delta t) - \vec{r}(t)}{\Delta t} = \vec{r}'(t) = \frac{d\vec{r}}{dt}. \]

In terms of coordinates:

\[ \vec{V}(t) = f'(t) \hat{i} + g'(t) \hat{j} + h'(t) \hat{k} = \frac{dx}{dt} \hat{i} + \frac{dy}{dt} \hat{j} + \frac{dz}{dt} \hat{k}. \]

Speed is defined as:

\[ \text{Speed} = || \vec{V}(t) || = \sqrt{f'(t)^2 + g'(t)^2 + h'(t)^2}. \]
\[ \vec{v}(t) \text{ is tangent to the path at each point:} \]

\[ \vec{r}(t+\Delta t) - \vec{r}(t) \]

\[ \begin{array}{c}
\vec{r}(t) \\
\vec{r}(t+\Delta t)
\end{array} \]

Becomes more and more tangent as \( \Delta t \to 0 \).

\[ \vec{v}(t) \text{ points in the direction of the motion.} \]

**Def:** Let \( \vec{r}(t) = f(t)i + g(t)j + h(t)k \) be the position vector of an object at time \( t \). The acceleration vector \( \vec{a}(t) \) is defined as

\[ \vec{a}(t) = \lim_{\Delta t \to 0} \frac{\vec{v}(t+\Delta t) - \vec{v}(t)}{\Delta t} = \frac{d\vec{v}}{dt} = \vec{v}'(t) = \vec{r}''(t) = \frac{d^2\vec{r}}{dt^2}. \]

In terms of coordinates:

\[ \vec{a}(t) = f''(t)i + g''(t)j + h''(t)k. \]

\( \vec{a}(t) \) reflects the changes in both direction and magnitude of the velocity \( \vec{v}(t) \).

**Ex.** A particle starts at the point \( P = (3, 2, -5) \) and moves along a straight line toward \( Q = (5, 7, -2) \) at a speed of \( 5 \text{ cm/sec} \). Let \( x, y, z \) be measured in cm.

(a) Find the particle's velocity vector.

(b) Find parametric equations for the particle's motion.

\[ \vec{V}(t) \parallel \vec{PQ} = 2i + 5j + 3k \]

\( \vec{v}(t) \) in seconds.

\[ \| \vec{V}(t) \| = 5 \text{ cm/sec}. \]

\[ \vec{v}(t) = c \cdot 2i + c \cdot 5j + c \cdot 3k \quad c = ? \]
\[ \mathbf{v} = \sqrt{4 \mathbf{c}^2 + 25 \mathbf{c}^2 + 9 \mathbf{c}^2} = \sqrt{38} \mathbf{c} = 5 \rightarrow c = \frac{5}{\sqrt{38}} \]

\[ \mathbf{v}(t) = \frac{5}{\sqrt{38}} \cdot \mathbf{q} = \frac{10}{\sqrt{38}} \mathbf{i} + \frac{25}{38} \mathbf{j} + \frac{15}{38} \mathbf{k} \]

\[ \mathbf{a}(t) = 0 \]

Indeed, neither the direction nor the magnitude of the velocity changes.

**Example:** Let \( \mathbf{r}(t) = (\cos t) \mathbf{i} + (\sin t) \mathbf{j} \). Find \( \mathbf{v}(t), \mathbf{a}(t) \), \( t \in [0, 2\pi] \)

\[ x = \cos t, \ y = \sin t, \ z = 0 \]

\[ \mathbf{v}(t) = -\sin(t) \mathbf{i} + \cos(t) \mathbf{j} \]

\[ \mathbf{a}(t) = -\cos(t) \mathbf{i} - \sin(t) \mathbf{j} \parallel \mathbf{v}(t) \]

\[ \mathbf{a}(t) \perp \mathbf{v}(t) \]

\[ || \mathbf{v}(t) || = 1, \quad \mathbf{a}(t) \] reflects changes in the direction of \( \mathbf{v}(t) \).

Length of \( C \) = \[ \int_0^{2\pi} || \mathbf{v}(t) || dt = \int_0^{2\pi} (\sin^2(t) + \cos^2(t)) dt = 2\pi. \]
Ex: Consider the motion of an object:

\[ \vec{r}(t) = t^2 \hat{j}, \quad t \geq 0 \]

Find \( \vec{v}(t) \) and \( \vec{a}(t) \). Describe the motion.

The object moves faster and faster along the y-axis.

\[ \vec{v}(t) = 2t \hat{j} \quad \text{parallel to the y-axis} \]
\[ s(t) = \sqrt{4t^2} = 2t \quad \text{speed increasing} \]
\[ \vec{a}(t) = 2 \hat{j} \quad \text{acceleration not \( \vec{v} \) as the magnitude of \( \vec{v}(t) \) changes.} \]
\[ \vec{a}(t) \] reflects changes in the magnitude of \( \vec{v}(t) \).

Let \( \vec{v}(t) \) be the velocity in a motion. Then

Distance traveled between \( t=a \) and \( t=b \)

\[ = \int_{a}^{b} \| \vec{v}(t) \| dt \]

Suppose an object moves along a curve \( C \), covers the curve once for \( t \) in \( [a,b] \), then the length of \( C \)

\[ \text{Length of } C = \int_{a}^{b} \| \vec{v}(t) \| dt \]
17.3 - 17.4 Vector Fields

A vector field on the plane is a function which to each point \((x, y)\) in some region prescribes a vector:

\[
(x, y) \rightarrow \vec{F}(x, y)
\]

In 3D, it is a function which to each \((x, y, z)\) prescribes a 3D vector \(\vec{F}(x, y, z)\):

\[
(x, y, z) \rightarrow \vec{F}(x, y, z)
\]

So:

When we sketch vector fields, we usually scale length.
Vector fields are very important: force fields, current velocity fields, etc.

We already know vector fields: if we have a function \(f(x, y)\), then we have its gradient vector field:

Given \(f(x, y)\), we have

\[
(x, y) \rightarrow \nabla f(x, y)
\]

Given \(f(x, y, z)\), we have

\[
(x, y, z) \rightarrow \nabla f(x, y, z)
\]
Ex: \( f(x,y) = x^2 \). Sketch the gradient vector field for \( f \).

\[ \nabla f(x,y) = 2x \mathbf{i} \]

\[ F(x,y) = 2x \mathbf{i} + 2y \mathbf{j} \]

Ex: Sketch the vector field \( F(x,y) = 2x \mathbf{i} + 2y \mathbf{j} \).

Observe that \( F(x,y) = \text{grad} f(x,y) \), where \( f(x,y) = x^2 + y^2 \).

Denote \( F = x \mathbf{i} + y \mathbf{j} \). Then \( F(x,y) \) can be written as \( \mathbf{F}(F) = 2 \mathbf{F} \).

\[ F(x,y) \parallel \mathbf{i} + \mathbf{j} \] is the position vector of \( (x,y) \).

\( F(x,y) \) at each point \( (x,y) \) point directly away from the origin. The magnitude increases as we move away from the origin.

If possible, we write vector fields in terms of \( \mathbf{r} = x \mathbf{i} + y \mathbf{j} \).

Def: A vector field \( \mathbf{F}(x,y) \) or \( \mathbf{F}(x,y,z) \) is called a gradient vector field if for some \( f(x,y) \) \(( f(x,y,z) \)):

\[ \mathbf{F}(x,y) = \nabla f(x,y) \quad \text{or} \quad \mathbf{F}(x,y,z) = \nabla f(x,y,z) \]
Perhaps every vector field is a gradient field?

\[ \mathbf{F}(x, y) = 2xy \mathbf{i} + xy \mathbf{j} \]

Suppose that for some \( f(x, y) \)
\[ \mathbf{F}(x, y) = \nabla f(x, y) = f_x(x, y) \mathbf{i} + f_y(x, y) \mathbf{j}, \]

Then \( f(x, y) \) has to be such that
\[ f_x(x, y) = 2xy, \quad f_y(x, y) = xy \]

\[ f(x, y) = x^2y + C(y) \]
\[ f_y(x, y) = x^2 + C'(y) \quad \text{not } xy \]

There is a simple way to test if a given vector field
\[ \mathbf{F}(x, y) = F_1(x, y) \mathbf{i} + F_2(x, y) \mathbf{j} \]
is a gradient field. If it is, then for some \( f(x, y) \):
\[ F_1(x, y) = f_x(x, y), \quad F_2(x, y) = f_y(x, y), \]

As \( f_{xy}(x, y) = f_{yx}(x, y) \), we have then:
\[ \frac{\partial F_1}{\partial y} = f_{xy} = f_{yx} = \frac{\partial F_2}{\partial x}. \]

So:
\[ \text{If } \frac{\partial F_1}{\partial y} \neq \frac{\partial F_2}{\partial x}, \text{ then } \mathbf{F} \text{ is not a gradient field.} \]

This is very important!
For \( F(x,y) = 2xy^2 + xy^2 \), we have:

\[
\frac{\partial}{\partial y} [2xy^2] = 2x, \quad \frac{\partial}{\partial x} [xy^2] = y
\]

So \( \frac{\partial F_1}{\partial y} \neq \frac{\partial F_2}{\partial x} \). Not a gradient field.