1) Let $f$ be defined in a neighborhood $N_{f}$ of $x_{0}$, $g$ be defined in a neighborhood $N_{g}$ of $x_{0}$. Then $f \cdot g$ is defined in $N_{f} \cap N_{g}$ which is a neighborhood of $x_{0}$. Let's examine the limit

$$\lim_{x \to x_{0}} \frac{(f \cdot g)(x) - (f \cdot g)(x_{0})}{x - x_{0}} = \lim_{x \to x_{0}} \frac{(f(x) - f(x_{0})) + (g(x) - g(x_{0}))}{x - x_{0}}$$

$$= \lim_{x \to x_{0}} \frac{f(x) - f(x_{0})}{x - x_{0}} + \lim_{x \to x_{0}} \frac{g(x) - g(x_{0})}{x - x_{0}} = f'(x_{0}) + g'(x_{0}).$$

The second equality holds as both limits $\lim_{x \to x_{0}} \frac{f(x) - f(x_{0})}{x - x_{0}}$ and $\lim_{x \to x_{0}} \frac{g(x) - g(x_{0})}{x - x_{0}}$ exist.

2) Let $a \in \mathbb{R}$ be given. We shall prove the following

**Lemma:** If $a = 0$, then

$$\lim_{x \to a} f(x) = f(a) = 0.$$

If $a \neq 0$, then

$$\lim_{x \to a} f(x)$$

does not exist.

(A "lemma" is a proposition helpful to prove a bigger theorem.)
Proof of the lemma: Let $a = 0$. Let a sequence $\{x_n\}$ be such that

$$x_n \to 0 \quad \text{as} \quad n \to +\infty \quad (1)$$

be given. We shall show that

$$f(x_n) \to 0 \quad \text{as} \quad n \to +\infty. \quad (2)$$

Take $\varepsilon > 0$. Let $N \in \mathbb{N}$ be such that

$$|x_n| < \varepsilon \quad \text{for} \quad n \geq N. \quad (3)$$

(Such $N$ exists from (1).) The definition of $f$ and (3) imply that

$$|f(x_n)| < \varepsilon \quad \text{for} \quad n \geq N.$$

Hence, (2). By the sequential definition of the limit we conclude that

$$\lim_{x \to 0} f(x) = 0 = f(0).$$

To prove the second part of the lemma take $a \neq 0$. Let $\{x_n\}_{n=1}^{\infty}$, $\{y_n\}_{n=1}^{\infty}$ be sequences such that $x_n \in \mathbb{Q}$, $y_n \in \mathbb{R} \setminus \mathbb{Q}$ for $n \in \mathbb{N}$, and $x_n \to a$, $y_n \to a$ as $n \to +\infty$.

Since $f(x_n) = x_n$, $f(y_n) = 0$ for $n \in \mathbb{N}$ we have:

$$f(x_n) \to a, \quad f(y_n) \to 0 \quad \text{as} \quad n \to +\infty.$$

By the sequential definition of the limit we conclude that the limit

$$\lim_{x \to a} f(x)$$

does not exist.
The lemma implies that \( f \) is not continuous at any point \( a \neq c \). Thus, \( f \) is not differentiable at any point \( a \neq 0 \).

Let's examine the existence of \( f'(0) \). For any \( x \neq 0 \) we have

\[
\frac{f(x) - f(0)}{x - 0} = \frac{f(x)}{x} = \begin{cases} 
1 & \text{if } x \text{ rational} \\
0 & \text{if } x \text{ irrational}
\end{cases}
\]

Thus the limit

\[
\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0}
\]

does not exist. We conclude that \( f \) is not differentiable at \( a = 0 \) either. (Although, \( f \) is continuous at \( a = 0 \).)

4) The product rule cannot be applied as \( h(x) = |x| \) is not differentiable at \( 0 \). The rest follows from 45 below: \( g'(0) \) exists and \( g'(0) = 101 = 0 \).

5) Since \( g \) is continuous at \( 0 \), \( g \) is defined in a nbhd of \( 0 \). Thus \( f \) is defined in a nbhd of \( 0 \). We have

\[
\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{xg(x) - 0g(0)}{x} = \lim_{x \to 0} g(x) = g(0).
\]

Thus \( f'(0) \) exists, and \( f'(0) = g(0) \).