3) Suppose that \( f \) is not constant. Then there exist \( x_1, x_2 \in \mathbb{R} \) such that \( f(x_1) \neq f(x_2) \). Without any loss of generality we may assume that \( x_1 < x_2 \). Since \( f \) is continuous in \([x_1, x_2]\) the Intermediate Value Theorem applies. Take \( \nu \) such that \( \nu \) is between \( f(x_1) \) and \( f(x_2) \) and \( \nu \in \mathbb{R} \setminus \mathbb{Q} \). By the IVT, there exists \( c \in [x_1, x_2] \) such that \( f(c) = \nu \). Contradiction as \( f \) takes only rational values and \( \nu \) is irrational.

4) Consider the function \( g(x) = f(x) - x \), \( x \in [0, 1] \).

If \( f(0) = 0 \), \( x = 0 \) is a fixed point. Suppose \( f(0) \neq 0 \).

Then \( f(0) > 0 \) as \( f \) takes values in \([0, 1]\), thus, \( g(0) > 0 \).

Since \( f(x) \leq 1 \) for all \( x \in [0, 1] \), in particular, \( f(1) \leq 1 \), we obtain \( g(1) \leq 0 \). \( g(x) \) is continuous in \([0, 1]\) as the difference of two continuous functions, and \( g(0) > 0 \), \( g(1) \leq 0 \). Thus

\[
g(1) \leq 0 < g(0).
\]

By the Intermediate Value Theorem, there exists \( x_0 \in [0, 1] \) such that \( g(x_0) = 0 \). But then \( f(x_0) = x_0 \) and \( x_0 \) is a fixed point.

5) Suppose \( L < N \). Take \( \varepsilon = \frac{M - L}{2} \). Since \( \lim_{n \to +\infty} a_n = L \), \( \lim b_n = M \), there exist \( N_1, N_2 \in \mathbb{N} \) such that

\[
|a_n - L| < \varepsilon \quad \text{for} \quad n \geq N_1, \\
|b_n - M| < \varepsilon \quad \text{for} \quad n \geq N_2.
\]
Let \( N = \max \{ N_1, N_2 \} \). The latter inequalities imply that for all \( n \geq N \) we have

\[ L - \varepsilon < a_n < L + \varepsilon \quad \text{and} \quad M - \varepsilon < b_n < M + \varepsilon. \]

But \( L + \varepsilon = \frac{L + M}{2} \), \( M - \varepsilon = \frac{L + M}{2} \). Hence, for \( n \geq N \),

\[ a_n < b_n. \]

Contradiction. Therefore, \( L \geq M \).

Example: Let \( a_n = \frac{1}{n} \), \( b_n = 0 \) for \( n = 1, 2, \ldots \). We have \( a_n > b_n \) for \( n = 1, 2, \ldots \), yet \( \lim_{n \to \infty} a_n = \lim_{n \to \infty} b_n = 0. \)

6) Let \( f(x) = x^2 \), \( I = (0, 1) \). \( f \) is continuous on \((0, 1)\), yet \( f \) does not attain its minimum or maximum values in \((0, 1)\).

For every \( x \in (0, 1) \), there exist \( y \in (0, 1) \) such that \( y > x \), thus \( f(y) > f(x) \). Hence, the largest value is not attained. Similarly, the smallest value is not attained. Obviously, the effect is due to the fact that the interval \((0, 1)\) is not closed.