2) We prove part (A). The proof of (B) is similar. Let $c \in \mathbb{R}$ be such that $f(c) > 0$. Take $\bar{E} = \frac{f(c)}{2}$. Then $\bar{E} > 0$. Since $f$ is continuous at $c$ in $I$, there exists a $\delta > 0$ such that:

$$|f(x) - f(c)| < \bar{E} \quad \text{provided} \quad 1 \cdot x - c \cdot 1 < \bar{\delta}, \ x \in I.$$  

Since $x$ is an interior point of $I$, for some $\delta > 0$, $(x-\delta, x+\delta) \subseteq I$. Without any loss of generality, we can assume $\bar{\delta} = \delta$. (If not, we replace $\delta$ with $\min\{\bar{\delta},\delta\}$.) Thus we have:

$$|f(x) - f(c)| < \bar{E} \quad \text{provided} \quad 1 \cdot x - c \cdot 1 < \bar{\delta}. \quad (1)$$

(1) implies, in particular:

$$f(c) - \bar{E} < f(x) \quad \text{for} \quad x \in (c-\bar{\delta}, c+\bar{\delta}).$$

Since $\bar{E} = \frac{f(c)}{2}$, we obtain:

$$f(x) > \frac{f(c)}{2} > 0 \quad \text{for} \quad x \in (c-\bar{\delta}, c+\bar{\delta}).$$

Therefore, (A) holds with $\varepsilon = \bar{\delta}$.

If $f(c) = 0$ we cannot conclude anything about the sign of $f$ in a nbhd of $c$.

3) Here we can use a direct proof or a proof by contradiction. Let's look at both.

**A direct proof:** We want to prove that $f(x) = 0$ for all $x \in \mathbb{R}$.

Take an arbitrary $x \in \mathbb{R}$. By Homework 5, Problem 3, we know that there exists a sequence of rationals convergent to $x$: $\{q_n\}_{n=1}^{\infty}$, $q_n \in \mathbb{Q}$ for $n \in \mathbb{N}$, $\lim_{n \to \infty} q_n = x$. Since $f$ is continuous, we obtain

$$\lim_{n \to \infty} f(q_n) = f(x).$$

But $f(q_n) = 0$. Thus, $\lim_{n \to \infty} f(q_n) = 0 = f(x)$. Since $x \in \mathbb{R}$ was arbitrary,

$$f(x) = 0 \quad \text{for all} \quad x \in \mathbb{R}.$$  

**An alternative proof by contradiction:** Suppose there exists $c \in \mathbb{R}$ such that $f(c) \neq 0$. Suppose first that $f(c) > 0$. Then by Prop H6.1 there exists a nbhd $(c-\varepsilon_0, c+\varepsilon_0)$ such that $f(x) > 0$ for all $x \in (c-\varepsilon_0, c+\varepsilon_0)$.

Contradiction as by the density of rationals $(c-\varepsilon_0, c+\varepsilon_0)$ contains rational numbers. We reason similarly in the case $f(c) < 0$.

By Prop H6.1, in some nbhd $(c-\varepsilon_0, c+\varepsilon_0)$, $f(x) < 0$ for all $x \in (c-\varepsilon_0, c+\varepsilon_0)$.

A contradiction as the nbhd contains rational numbers and $f(q) = 0$ for all $q \in \mathbb{Q}$.
4) \( f(x) = \frac{1}{\sqrt{x}} \) is defined in \((0, +\infty)\). Take an \( \varepsilon > 0 \).

Take a \( p \) such that \( p > \frac{1}{\varepsilon^2} \). Then:

\[
| \frac{1}{\sqrt{x}} - 0 | = \frac{1}{\sqrt{x}} < \varepsilon \quad \text{whenever} \quad x > p.
\]

Indeed, \( x > p \) gives \( x > \frac{1}{\varepsilon^2} \), which in turn implies \( \frac{1}{\sqrt{x}} < \varepsilon \).

Thus, by Def HC.1, \( \lim_{x \to +\infty} \frac{1}{\sqrt{x}} = 0 \).

5) We use Def HC.1. Let \( \varepsilon > 0 \) be given. As \( \lim_{x \to +\infty} f(x) = L \)

\[
\lim_{x \to +\infty} g(x) = M,
\]
then exist \( p_1 > 0 \), \( p_2 > 0 \) such that

\[
1. \quad | f(x) - L | < \frac{\varepsilon}{2} \quad \text{for} \quad x > p_1 \quad (2)
\]

\[
2. \quad | g(x) - M | < \frac{\varepsilon}{2} \quad \text{for} \quad x > p_2 \quad (3)
\]

Take \( p = \max \{ p_1, p_2 \} \). Then for \( x > p \) (2) and (3) hold. We obtain:

\[
| f(x) + g(x) - (L + M) | \leq | f(x) - L | + | g(x) - M | < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \quad \text{for} \quad x > p \quad (4)
\]

Since \( \varepsilon \) was arbitrary, \( \lim_{x \to +\infty} (f(x) + g(x)) = L + M \) by (4).

6) \underline{Def HC.2} : Let \( f \) be defined in \((-\infty, a)\) for some \( a \in \mathbb{R} \).

Let \( L \in \mathbb{R} \). We say that

\[
\lim_{x \to -\infty} f(x) = L
\]

\[
\forall \varepsilon > 0 \quad \exists p > a \quad \forall x \in (-\infty, a) \quad (x \leq p \Rightarrow | f(x) - L | < \varepsilon).
\]