1) Suppose that $L > M$. Take $E_0 = \frac{L - M}{2}$. Since $L > M$, $E_0 > 0$. Thus, there exist $N_1, N_2 \in \mathbb{N}$ such that

\[ a_n - L < E_0 \quad \text{for} \quad n \geq N_1 \]
\[ b_n - M < E_0 \quad \text{for} \quad n > N_2 \]

Take $N = \max\{N_0, N_1, N_2\}$. Then for all $n \geq N$ we have:

\[ a_n - L < E_0, \quad b_n - M < E_0, \quad a_n < b_n. \]

In particular, for $n = N$ we obtain:

\[ L - E_0 < a_N, \quad a_N < M + E_0, \quad a_N \leq b_N. \tag{1} \]

Since $E_0 = \frac{L - M}{2}$, $L - E_0 = \frac{L + M}{2}$, $M + E_0 = \frac{L + M}{2}$. (1) gives

\[ a_N > \frac{L + M}{2} > b_N, \quad a_N \leq b_N. \]

Contradiction. Thus, $L \leq N$.

2) Take $\{a_n\}_{n=1}^{\infty}$, $a_n = -1$ for all $n \in \mathbb{N}$. Take $\{c_n\}_{n=1}^{\infty}$, $c_n = 1$ for all $n \in \mathbb{N}$.

\[ a_n, c_n \text{ are constant sequences and therefore, they are Cauchy sequences.} \]

Let $\{b_n\}_{n=1}^{\infty}$ be the sequence $b_n = (-1)^n$, $n \in \mathbb{N}$. Then

\[ a_n \leq b_n \leq c_n \quad \text{for} \quad n = 1, 2, \ldots \]

$\{b_n\}_{n=1}^{\infty}$ is divergent as proved in class. Thus, $\{b_n\}_{n=1}^{\infty}$ is not Cauchy.

3) Since rationals and irrationals are dense in $\mathbb{R}$ for every $n = 1, 2, \ldots$ there exist $q_n \in \mathbb{Q}$, $p_n \in \mathbb{R} \setminus \mathbb{Q}$ such that:

\[ q < q_n < a + \frac{1}{n}, \quad a < q_n < a + \frac{1}{n}. \]

As $\lim_{n \to \infty} a = \lim_{n \to \infty} (a + \frac{1}{n}) = a$ the Squeeze Theorem gives

\[ \lim_{n \to \infty} q_n = a \quad \text{and} \quad \lim_{n \to \infty} p_n = a. \]

4) $\lim_{x \to 0} x^2 \cos \frac{1}{x} = 0$. Indeed, $x^2 \cos \frac{1}{x}$ is defined for all $x \neq 0$.

Thus $x^2 \cos \frac{1}{x}$ is defined in a deleted nbhd $N_\delta$ of $0$.

Let $\varepsilon > 0$ be given. Take $\delta = \sqrt{\varepsilon}$. Then for every $x \in N_\delta$:

\[ |x| \leq 1 \times 2^2 = 1 \times 1^2 < \varepsilon \quad \text{provided} \quad |x| < \delta. \]

5) Let $c \in \mathbb{R}$ be given. By Problem 3 there exist sequences

$\{q_n\}_{n=1}^{\infty}$, $\{p_n\}_{n=1}^{\infty}$ such that $q_n \in \mathbb{Q}$, $p_n \not\in \mathbb{Q}$ for $n \in \mathbb{N}$,

$\lim_{n \to \infty} q_n = \lim_{n \to \infty} p_n = c$. As $D(q_n) = 1$, $D(p_n) = 0$ for $n \in \mathbb{N}$,

$\lim_{n \to \infty} D(p_n) = 0 \neq \lim_{n \to \infty} D(q_n) = 1$. Thus, by Heine's characterization

$\lim_{x \to c} D(x)$ does not exist.