3) Prove Prop 16, page 45.

To prove that

\[ \bigcap_{F \in \mathcal{E}} F \neq \emptyset \]  \hspace{1cm} (1)

suppose the contrary. That is, suppose \( \bigcap_{F \in \mathcal{E}} F = \emptyset \).

Then \( \sim \bigcap_{F \in \mathcal{E}} F = R = \bigcup_{F \in \mathcal{E}} \sim F \). Since all sets \( F \) are closed, all sets \( \sim F \) are open. Hence, \( \{ \sim F : F \in \mathcal{E} \} \) is an open cover for \( R \). Let \( F_0 \in \mathcal{E} \) be bounded. Since \( F_0 \subseteq R \), \( \{ \sim F : F \in \mathcal{E} \} \) is an open cover for \( F_0 \). \( F_0 \) is closed and bounded.

From the Heine-Borel Theorem, there exists a finite subcover \( \{ \sim F_1, \sim F_2, \ldots, \sim F_k \} \) for \( F_0 \):

\[ F_0 \subseteq \bigcup_{i=1}^{k} \sim F_i \]  \hspace{1cm} (2)

(2) implies easily that \( F_0 \cap \sim F_1 \cap \ldots \cap \sim F_k = \emptyset \). Indeed, if \( x \in F_0 \cap \sim F_1 \cap \ldots \cap \sim F_k \), then \( x \in F_0 \) and \( x \notin \sim F_i \) for \( i = 1, 2, \ldots, k \). Contradiction with (2). Thus, \( F_0 \cap \sim F_1 \cap \ldots \cap \sim F_k = \emptyset \) which contradicts the assumption that every finite subcollection of \( \mathcal{E} \) has a nonempty intersection.

Thus, (1).

1) Yes, \( A \sim B \in \mathcal{A} \); that is \( \mathcal{A} \) is closed with respect to the difference of sets. Note first that \( \mathcal{A} \) is closed with respect to the intersection.

Let \( C, D \in \mathcal{A} \). Then \( \sim C, \sim D \in \mathcal{A} \) and \( \sim C \cap \sim D = \sim (C \cap D) \in \mathcal{A} \). The latter gives \( \sim (\sim (C \cap D)) = C \cap D \in \mathcal{A} \). Hence, \( \mathcal{A} \) is closed with respect to the intersection. But \( A \sim B = A \cap \sim B \). Thus, \( A \sim B \in \mathcal{A} \).
5) (c) of Th 8.1. Let
\[ C_1 = \{(a, b) : a < b, a, b \in \mathbb{R}\}, \quad C_2 = \{(a, b) : a < b, a, b \in \mathbb{R}\}. \]

We proved in class that \( B = \mathcal{D}(C_1) \). To prove that
\[ B = \mathcal{D}(C_2), \quad (3) \]
we use Prop 8.1. We have to show that
\[ C_1 \subseteq \mathcal{D}(C_2) \quad \text{and} \quad C_2 \subseteq \mathcal{D}(C_1). \quad (4) \]

Take any \((a, b) \in C_1\). Then
\[ (a, b) = \bigcup_{n=1}^{\infty} [a + \frac{1}{n}, b - \frac{1}{n}]. \]
As \([a + \frac{1}{n}, b - \frac{1}{n}] \in C_2\), \((a, b) \in \mathcal{D}(C_2)\) and the first part of (4) is proved. To prove the second part, take any \([a, b] \in C_2\).

Then
\[ [a, b] = \bigcap_{n=1}^{\infty} (a - \frac{1}{n}, b + \frac{1}{n}). \]
Since \((a - \frac{1}{n}, b + \frac{1}{n}) \in C_1\) and \(\mathcal{D}(C_1)\) as any \(\mathcal{D}\)-algebra is closed with respect to countable intersections, \([a, b] \in \mathcal{D}(C_1)\).

Hence, (4). By Prop 8.1, \(\mathcal{D}(C_1) = \mathcal{D}(C_2)\).

6) (d) of Th 8.1. Let
\[ C_3 = \{(a, +\infty) : a \in \mathbb{R}\}. \]
It suffices to show that
\[ C_3 \subseteq \mathcal{D}(C_1) \quad \text{and} \quad C_1 \subseteq \mathcal{D}(C_3). \quad (5) \]
Indeed, (5) via Prop 8.1 gives \(\mathcal{D}(C_3) = \mathcal{D}(C_1)\). Since \(\mathcal{D}(C_1) = B\), \(\mathcal{D}(C_3) = B\).

To prove (5), observe that for any \(a \in \mathbb{R}\):
\[ (a, +\infty) = \bigcup_{n=1}^{\infty} (a, n). \]
Thus, \(C_3 \in \mathcal{D}(C_1)\). To prove the second part of (5), take \((a, b) \in C_1\).

We have \((a, +\infty) \in \mathcal{D}(C_3)\) and \((b - \frac{1}{n}, +\infty) \in \mathcal{D}(C_3)\) for all \(n = 1, 2, \ldots\).

Thus, \(\cap (b - \frac{1}{n}, +\infty) = (\cap \cap b - \frac{1}{n}) = (a + \frac{1}{n}) \cap \mathcal{D}(C_3)\) for \(n = 1, 2, \ldots\). Then
\[ (a, +\infty) \cap (b - \frac{1}{n}, +\infty) = \mathcal{D}(C_3)\] for \(n = 1, 2, \ldots\), which gives \((a, b) = \bigcup_{n=1}^{\infty} (a, b - \frac{1}{n}) \in \mathcal{D}(C_3)\). Thus, (5).