1) From the assumptions, we have:
\[ a_n \leq a_{n+1}, \quad b_{n+1} \leq b_n \quad \text{for all } n=1,2,\ldots. \quad (1) \]
Also
\[ \lim_{n \to +\infty} (b_n - a_n) = 0. \quad (2) \]
As \( L_n, b_n \in [a_1, b_1] \) for all \( n=1,2,\ldots \), both sequences \( <a_n>_{n=1}^\infty \) and \( <b_n>_{n=1}^\infty \) are bounded and, by (1), monotone. Hence, they are convergent and:
\[ \lim_{n \to +\infty} a_n = a, \quad a = \sup \{ a_n : n \in N \}, \quad (3) \]
\[ \lim_{n \to +\infty} b_n = b, \quad b = \inf \{ b_n : n \in N \}. \quad (4) \]
By (2), (3), and (4):
\[ \lim_{n \to +\infty} (b_n - a_n) = \lim_{n \to +\infty} b_n - \lim_{n \to +\infty} a_n = b - a = 0. \]
Thus \( a = b \). Let \( c = a = b \). We shall prove that
\[ \bigcap_{n=1}^\infty [a_n, b_n] = \{ c \}. \quad (5) \]
By (3) and (4) we have:
\[ a_n \leq c \leq b_n \quad \text{for } n=1,2,\ldots. \]
Thus, \( \{ c \} \subseteq \bigcap_{n=1}^\infty [a_n, b_n] \). Suppose that for some \( d \in R \), \( d \neq c \), \( d \in \bigcap_{n=1}^\infty [a_n, b_n] \). Let \( |c - d| = E > 0 \). From (2), for some \( N \in N \), \( b_n - a_n < E \) whenever \( n \geq N \). But \( c, d \in [a_n, b_n] \) for all \( n \in N \), which implies \( b_n - a_n \geq E \) for all \( n \in N \). Contradiction.
Hence, (5).

2) Suppose \( E \) is countable. Let \( f : N \to E \) be a bijection. Denote \( s_n = f(n) \) for \( n=1,2,\ldots \). Then:
\[ E = \{ s_1, s_2, \ldots, s_n, \ldots \} \]
where each \( s_n \) is a sequence of 0's and 1's.
$s_1 = <a_1, a'_1, a''_1, \ldots, a''', \ldots >$

$s_2 = <a_2, \ldots, a'_2, \ldots >$

$s_n = <a_1, a'_1, a''_1, \ldots, a''_n, \ldots >$

Define a sequence $b = <q_i>_{i=1}^\infty$ as follows:

$q_i = \begin{cases} 0 & \text{if } a_i = 1 \\ 1 & \text{if } a_i = 0 \end{cases}$

Then $b \notin s_n$ for every $n \in \mathbb{N}$ and $b \in E$. Consequently,

Thus $E$ is not countable.

3) Let a set $X$ and a collection $C \subseteq \mathcal{P}(X)$ be given.

We shall prove that

$$\bigcup_{A \in C} A = \bigcap_{A \in C} \sim A \quad (6)$$

(The other part is similar.)

Let $x \in \bigcup_{A \in C} A$. Then $x \in \cup_{A \in C} A$. Hence, $x \in A$ for every $A \in C$, which gives $x \in \sim A$ for every $A \in C$. Thus, $x \in \bigcap_{A \in C} \sim A$ and the inclusion $\bigcup_{A \in C} A \subseteq \bigcap_{A \in C} \sim A$ is proved.

Let $x \in \bigcap_{A \in C} \sim A$. Then $x \in \sim A$ for every $A \in C$, which implies $x \in A$ for every $A \in C$. The latter gives $x \in \cup_{A \in C} A$ and hence $x \in \bigcup_{A \in C} A$. Therefore, $\bigcap_{A \in C} \sim A \subseteq \bigcup_{A \in C} A$ and (6) is proved.

4) Skipped
5) Let \( f : \mathbb{R} \rightarrow \mathbb{R} \) be defined as
\[
f(x) = 0 \quad \text{for all } x \in \mathbb{R}.
\]
Take \( A = (-1, 0) \), \( B = (0, 1) \). Then \( f(A \cap B) = f(\emptyset) = \emptyset \).
Yet, \( f(A) = f(B) = \{0\} \). Thus \( f(A) \cap f(B) = \{0\} \neq \emptyset \).

6) We prove (c) , (b) and (a) are similar.

Let \( B \subseteq \mathbb{X} \). Let \( x \in f^{-1}(\sim B) \). Then by the definition of the preimage, \( f(x) \in \sim B \). Hence, \( f(x) \notin B \) and \( x \notin f^{-1}(B) \).
Thus \( x \in \sim f^{-1}(B) \) and the inclusion:
\[
f^{-1}(\sim B) \subseteq \sim f^{-1}(B)
\]
is proved. To prove the opposite inclusion, take \( x \in \sim f^{-1}(B) \).
Then \( x \notin f^{-1}(B) \) which implies \( f(x) \notin B \). Hence, \( f(x) \in \sim B \)
and \( x \in f^{-1}(\sim B) \). Therefore, \( \sim f^{-1}(B) \subseteq f^{-1}(\sim B) \) and
(c) is proved.