Th 17.1 (Egoroff) Let \( f, f_n : D \to \mathbb{R}, n = 1, 2, \ldots \) be measurable, \( m(D) < +\infty \). Assume that \( f_n \to f \) a.e. on \( D \). Then for every \( \varepsilon > 0 \) there exists \( E_\varepsilon \subset D \) such that \( E_\varepsilon \in \mathcal{M}, \ m(E_\varepsilon) < \varepsilon \) and \( f_n \Rightarrow f \) on \( D \setminus E_\varepsilon \).

Before we prove the theorem recall the following two propositions:

(P1) If \( f \) is measurable, then \( |f| \) is measurable.

Indeed, \( |f| = f^+ + f^- \), \( f^+, f^- \) are measurable.

(By \( \#5, \PageIndex{7}, \))

(P2) If \( A_1, A_2, \ldots, A_n, \ldots \in \mathcal{M}, A_{n+1} \supseteq A_n \) for \( n = 1, 2, \ldots \)

then

\[ m\left( \bigcup_{n=1}^{\infty} A_n \right) = \lim_{n \to +\infty} m(A_n). \]

(By \( \#2, \PageIndex{6}, \))

Proof of Th 17.1: Let \( \varepsilon > 0 \) be fixed. We want to show that there exists \( E_\varepsilon \subset D \) such that

\[ m(E_\varepsilon) < \varepsilon \quad \text{and} \]

\[ \forall \varepsilon > 0 : \quad \forall N \in \mathbb{N} : \quad \forall n \geq N : x \in D \setminus E_\varepsilon \]

\[ |f_n(x) - f(x)| < \varepsilon. \quad (1) \]

In other words, \( f_n \Rightarrow f \) on \( D \setminus E_\varepsilon \).
Let \( D_0 \subseteq D \) be:
\[ D_0 = \{ x \in D : f_n(x) \neq f(x) \} \]
Then \( D_0 \in \mathcal{M} \), \( m(D_0) = 0 \). For each pair \( n \in \mathbb{N} \) and \( i \in \mathbb{N} \) define the following set:
\[ E_{i,n} = \{ x \in D \setminus D_0 : |f_m(x) - f(x)| < \frac{1}{2^i} \text{ for } m \geq n \} \quad (2) \]
As \( |f_m - f| \) is measurable, \( E_{i,n} \in \mathcal{M} \) for \( i, n \in \mathbb{N} \).

From (2), \( E_{i,n+1} \supseteq E_{i,n} \) for all \( i, n \in \mathbb{N} \). Also for every \( i \in \mathbb{N} \):
\[ \bigcup_{n=1}^{\infty} E_{i,n} = D \setminus D_0 \quad (3) \]
To prove (3), fix \( i \) and take \( x \in D \setminus D_0 \). Then
\[ f_n(x) \to f(x), \text{ thus, } |f_n(x) - f(x)| < \frac{1}{2^i} \text{ for } n \geq N \]
for some \( N \) (which depends on \( x \) and \( i \)). Hence
\[ x \in E_{i,N} \text{ and the inclusion } D \setminus D_0 \subseteq \bigcup_{n=1}^{\infty} E_{i,n} \text{ is proved.} \]

The opposite inclusion follows from (2). Thus, (3) is proved.

By (P2), for every fixed \( i = 1, 2, \ldots, \) we have:
\[ \lim_{n \to +\infty} m(E_{i,n}) = m(D \setminus D_0) = m(D) - m(D_0) = m(D) \quad (4) \]
(The second equality follows from Prop 12.3 as \( m(D) < +\infty \).)

From (4), for every fixed \( i \), there exists \( N_i \in \mathbb{N} \) such that
\[ m(E_{i,N_i}) > m(D) - \frac{E_i}{2^i}, \quad m(D \setminus E_{i\cap N_i}) < \frac{E_i}{2^i} \quad (5) \]
Define

\[ E\varepsilon = D \cup \bigcap_{i=1}^{\infty} E_{i, N_i} = \bigcup_{i=1}^{\infty} (D \cup E_{i, N_i}) \] 

By the latter equality and (5), we have

\[ m(E\varepsilon) = m\left( \bigcup_{i=1}^{\infty} (D \cup E_{i, N_i}) \right) \leq \sum_{i=1}^{\infty} \frac{\varepsilon_i}{2^i} = \varepsilon. \]

It remains to show that \( f_n \to f \) on \( D \cup E\varepsilon \). Observe that by (6):

\[ D \cup E\varepsilon = \bigcap_{i=1}^{\infty} E_{i, N_i}. \]

Therefore, for every \( x \in D \cup E\varepsilon \) we have

\[ x \in E_{i, N_i} \quad \text{for } i=1, 2, \ldots. \]

From (2), we obtain that for all \( x \in D \cup E\varepsilon \) and all \( i=1, 2, \ldots \):

\[ |f_m(x) - f(x)| < \frac{1}{2^i} \quad \text{for } m \geq N_i. \]  

We are ready to show (1). Let \( \eta > 0 \) be given. Let \( i_0 \in \mathbb{N} \) be such that \( \frac{1}{2^{i_0}} < \eta \). (7) implies that

\[ |f_n(x) - f(x)| < \frac{1}{2^{i_0}} < \eta \quad \text{for all } n \geq N_{i_0}, x \in D \cup E\varepsilon. \]

Hence, (1) holds with \( N_\varepsilon = N_{i_0} \). The proof is complete.