Recall: \( f \) is integrable over \( E \) if
\[
\int_E f = \int_E f^+ - \int_E f^-
\]
is defined and finite.

**Prop 23.1**: Let \( f \) be a measurable function defined on a measurable set \( E \). Then \( f \) is integrable in \( E \) if and only if \( |f| \) is integrable in \( E \).
If they are both integrable, then:
\[
\left| \int_E f \right| \leq \int_E |f|. \quad (1)
\]

**Proof**: Assume that \( f \) is integrable on \( E \). Then, by Remark 22.1, \( f^+ \) and \( f^- \) are integrable on \( E \). Thus by Th 21.1 (ii), \( f^+ + f^- \) is integrable on \( E \). But
\[
|f| = f^+ + f^- \quad (2)
\]
Hence, \( |f| \) is integrable. Assume that \( |f| \) is integrable on \( E \).

From (2), \( 0 \leq f^+ \leq |f| \). Hence, by Th 21.1:
\[
0 \leq \int_E f^+ \leq \int_E |f| < +\infty
\]
Thus \( f^+ \) is integrable. Similarly we obtain that \( f^- \) is integrable and so is \( f \) by Remark 22.1.

(1) follows from Th 22. (iii) and (i). Indeed:
\[
-\int f \leq f \leq \int |f|
\]
Thus
\[
-\int |f| \leq \int f \leq \int |f|
\]
The latter implies (1).

Therefore, for the Lebesgue integral integrability and absolute integrability are equivalent. It is not so for the Riemann integral.
**Ex:** Define $f : [0, 1] \rightarrow \mathbb{R}$ as
$$f(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ -1, & x \in [0, 1] \setminus \mathbb{Q} \end{cases}.$$ $f$ is not Riemann integrable in $[0, 1]$ but $|f| \equiv 1$ is. $f$ and $|f|$ are both Lebesgue integrable on $[0, 1]$ with the integral 0.

**Ex:** Consider $f : [1, +\infty) \rightarrow \mathbb{R}$ defined as:
$$f(x) = \frac{(-1)^{n+1}}{n}, \quad x \in [n, n+1), \quad n = 1, 2, \ldots.$$ It is easy to prove that
$$\lim_{x \to +\infty} \int_{n}^{n+1} f(x) \, dx = \sum_{n=1}^{+\infty} \frac{(-1)^{n+1}}{n} < +\infty.$$ Yet, $\int_{1}^{+\infty} f(x) \, dx = \sum_{n=1}^{+\infty} \frac{1}{n} = +\infty$. Thus, $|f|$ is not Lebesgue integrable in $[1, +\infty)$ and $f$ is not either. Hence, for functions which are not necessarily non-negative, summability in the sense of Riemann and summability in the sense of Lebesgue are different.

**Prop 23.2** Let $f$ be integrable over $E$, $g$ be measurable and defined on $E$. If
$$|g| \leq f \text{ a.e. on } E,$$
then $g$ is integrable on $E$.

**Proof:** By Th 21.1:
$$0 \leq \int_{E} |g| \leq \int_{E} f < +\infty.$$ So $|g|$ is integrable and, by Prop 23.1, so is $g$.

**Lemma 23.1:** Let $\{a_n\}_{n=1}^{\infty}$ be a sequence from $\mathbb{R}$, $c \in \mathbb{R}$. Then
(a) $\lim (c - a_n) = c - \lim a_n$ \quad \text{(b)} $\lim (c + a_n) = c + \lim a_n$.

**Proof:** Easy from Th 14.3.
Theorem 23.1 (The Lebesgue Dominated Convergence Theorem): Let \( g \) be integrable over \( E \). Let \( f_n, n=1,2,\ldots, \) be measurable functions defined on \( E \) such that:

\[
1_{f_n} \leq g \quad \text{on } E \quad \text{for } n=1,2,\ldots \quad (3)
\]

Assume that for some function \( f \) defined on \( E \) we have:

\[
f_n \rightarrow f \quad \text{a.e. on } E \quad \quad (4)
\]

Then \( f, f_n, n=1,2,\ldots \), are integrable on \( E \) and

\[
\int_E f = \lim_{n \to \infty} \int_E f_n. \quad (5)
\]

**Proof:** \( f \) is measurable as an a.e. limit of measurable functions.

By Prop. 23.2, \( f_n \) are all integrable on \( E \) for \( n=1,2,\ldots \). Observe that \( 1_{f_n} \rightarrow 1_f \) a.e. on \( E \) by (4). Thus from (3):

\[
1_f \leq g \quad \text{a.e. on } E.
\]

By Prop 23.2, \( f \) is integrable on \( E \). To prove (5) observe that by (3):

\[
-g \leq f_n \leq g \quad \text{on } E \quad \text{for } n=1,2,\ldots
\]

Hence:

\[
g - f_n \geq 0 \quad \text{and} \quad f_n + g \geq 0 \quad \text{on } E \quad \text{for } n=1,2,\ldots \quad (6)
\]

We have also:

\[
g - f_n \rightarrow g - f \quad \text{a.e. on } E \quad , \quad f_n + g \rightarrow f + g \quad \text{a.e. on } E \quad (7)
\]

By (6) and (7) we can apply Fatou's Lemma to both sequences.

We obtain:

\[
\int_E (g - f) \leq \liminf_{n \to \infty} \int_E (g - f_n) = \lim_{n \to \infty} \int_E (g - f), \quad (\text{Th 22.1, F.L.})
\]

\[
\int_E (g - f) = \int_E g - \int_E f \leq \lim_{n \to \infty} \int_E (g - f_n) = \lim_{n \to \infty} \int_E (g - f_n).
\]
From Lemma 23.1:

\[ \int_E q - \int_E f \leq \int_E q - \lim_{n \to \infty} \int_E f_n. \]

Thus:

\[ \int_E f \geq \lim_{n \to \infty} \int_E f_n. \quad (8) \]

Applying Fatou's Lemma to \( g + f_n \), we obtain:

\[ \int_E (g + f) = \int_E g + \int_E f \leq \lim_{n \to \infty} \int_E (g + f_n) = \lim_{n \to \infty} (\int_E g + \int_E f_n). \]

By Lemma 23.1:

\[ \int_E g + \int_E f \leq \int_E g + \lim_{n \to \infty} \int_E f_n. \]

Thus:

\[ \int_E f \leq \lim_{n \to \infty} \int_E f_n. \quad (9). \]

(9) and (8) give:

\[ \int_E f \leq \lim_{n \to \infty} \int_E f_n \leq \lim_{n \to \infty} \int_E f_n \leq \int_E f. \]

The latter implies (5).

\[ \square \]

**Prop 23.3**: Th 23.1 remains valid for extended real-valued functions \( f, f_n, g, n = 1, 2, \ldots \) with the assumption

\[ |f_n| \leq g \text{ on } E \text{ for } n = 1, 2, \ldots \]

replaced by

\[ |f_n| \leq g \text{ a.e. on } E. \]

\[ \triangleq \]